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Reciprocity Theorem

Consider inverse nuclear reaction :

$$x + X \rightleftharpoons Y + y$$

to find relation between $\sigma(\alpha \to \beta)$ and $\sigma(\beta \to \alpha)$

under equilibrium condition, the number of channels in the energy range dE are equal i.e.

$$\boxed{N_\alpha = N_\beta}$$

In phase space, number of particles in volume V (where reaction occurs) is given by :

$$\boxed{N = \frac{4\pi V p^2 dp}{h^3}}$$

since $E = \frac{p^2}{2m} \gg dE = \frac{p\,dp}{m}$

$$= \frac{mv\,dp}{m} = v\,dp$$

$\therefore$ $E \gg v\,dp$

$v\,dp \gg dp = \frac{dE}{v}$

$$N = \frac{4\pi V p^2 dE}{8\pi^3 \hbar^3 v} = \frac{V p^2 dE}{2\pi^2 \hbar^3 v}$$

$$N = \frac{V p^2 dE}{8\pi^2 \hbar^3 v}$$ final momentum can be different only if energy is d

for $N_\alpha = N_\beta$ (dE is same), we write

$$p_\alpha^2 v_\beta = p_\beta^2 v_\alpha$$

from Fermi Golden Rule; $\boxed{W(\alpha,\beta) = \text{inc flux} \; \sigma(\alpha\to\beta)}$

W – transition probability with 'σ' cross section.

$\because$ Incd. flux $\phi = \frac{v}{V}$ (v velocity of particle)

Applying respective transition probability in eq.,

$$p_\alpha^2 v_\beta W(\alpha\to\beta) = p_\beta^2 v_\alpha W(\beta\to\alpha)$$

$$p_\alpha^2 v_\beta \sigma(\alpha\to\beta) \frac{v_\alpha}{V} = p_\beta^2 v_\alpha \sigma(\beta\to\alpha) \frac{v_\beta}{V}$$

or; $$p_\alpha^2 \sigma(\alpha\to\beta) = p_\beta^2 \sigma(\beta-\alpha)$$

Since; $p_\alpha = k_\alpha \hbar$. So,

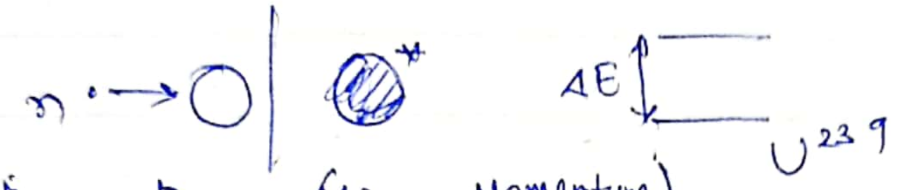
$$k_\alpha^2 \sigma(\alpha\to\beta) = k_\beta^2 \sigma(\beta\to\alpha)$$

This is the necessary condition for inverse nuclear reaction called Reciprocity theorem.

Resonance in nuclear React^n:

$$n + {}^{238}U \rightarrow {}^{239}U^* \qquad {}^{238}U(n,\gamma)\,{}^{239}U^*$$

$$Q = m_n c^2 + m({}^{238}U)c^2 - m({}^{239}U)c^2 = 4.86\,\text{MeV}$$



$P_n = P_{U239}$ (linear Momentum conserved)

$$(KE)_n + (RE)_n + (RE)_{U238} = (RE)_{U^{239}} + (K)_{U^{239}} + \Delta E$$

$$K_n + m_n c^2 + m c^2_{238} = (RE)_{U^{239}} + K_{U^{239}} + \Delta E$$

$$K_n + Q = K_{^{239}U} + \Delta E = \frac{p^2_{239}}{2m_{U^{239}}} + \Delta E = \frac{p_n^2}{2m_n}\cdot\frac{m_n}{m_{U^{239}}} + \Delta E$$

$$= K_n \cdot \frac{m_n}{M_Y} + \Delta E$$

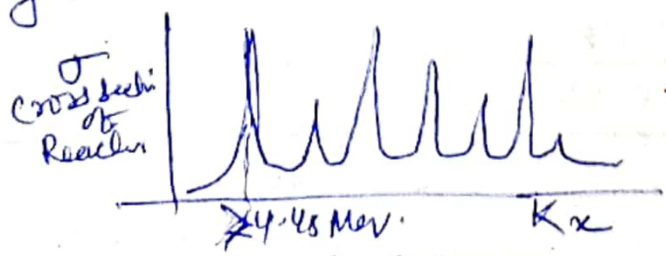
$$K_x\left(1 - \frac{m_x}{M_Y}\right) = \Delta E - Q$$

If $K_x \to 0$ {projectile of almost zero energy}

$\Delta E = Q = 4.49$ MeV (here) → Which may not be allowed energy level of Y^*
→ Discrete energies

or; $\boxed{\Delta E = Q + K_x\left(1 - \frac{m_x}{M_Y}\right)}$

⇒ If K_x or energy of projectile is such that ΔE matches with Y^* the probability is high to get absorbed.



Resonance

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Resonance scattering : Breit Wigner Dispersion formula:

* This formula examines the behavior of cross section near resonance

A formula which relates the C.S.e of a particular nuclear reaction with the energy of the incident particle is called B.W-formula.

* Cross section total for reaction:

$$\sigma_T = \frac{4\pi}{k^2}\sin^2\delta_\ell \; ; \; \delta_\ell = \pi/2 \quad (1)$$

δ_ℓ — phase shift of ℓ^{th} partial scattering.

* for Resonance at $E = E_0$

$$\left.\begin{aligned} &\text{if } \delta_\ell(E_0) = \pi/2 \\ &\text{then } \sin\delta_\ell(E_0) = 1 \\ &\cos\delta_\ell(E_0) = 0 \end{aligned}\right\} \quad (2)$$

Taylor Series;

$$\left[f(x)\right]_{x=x_0} = f(x_0) + (x-x_0)\left.\frac{\partial F}{\partial x}\right|_{x=x_0} + \cdots \quad (3)$$

then:

$$\left[\sin\delta_\ell(E_0)\right]_{E=E_0} = \sin\delta_\ell(E_0) + (E-E_0)\left.\left[\cos\delta_\ell(E_0)\frac{\partial\delta_\ell(E)}{\partial E}\right]\right|_{E=E_0} \quad (3')$$

Substituting (2) in (3').

$$\left[\sin\delta_\ell(E_0)\right]_{E=E_0} = 1 + (E-E_0)\left.\left[\underbrace{\cos\delta_\ell(E_0)}_{=0}\frac{\partial\delta_\ell(E)}{\partial E}\right]\right|_{E=E_0}$$

Similarly; $\boxed{\sin\delta_\ell(E_0) = 1}$

$$\cos\delta_\ell(E_0)\Big|_{E=E_0} = \cos\delta_\ell(E_0) + (E-E_0)\left(-\sin\delta_\ell(E_0)\frac{\partial\delta_\ell(E)}{\partial E}\right)\Big|_{E=E_0}$$

$$= 0 + -(E-E_0)\left.\frac{\partial\delta_\ell(E)}{\partial E}\right|_{E=E_0}$$

$$= \frac{2}{\Gamma}(E - E_0) \quad \text{—} \quad (4)$$

$$\Rightarrow \boxed{\frac{2}{\Gamma} = \frac{\partial}{\partial E}\delta_l(E)\Big|_{E=E_0}} \quad \text{—} \quad (4')$$

Rate of change of phase shift at resonance i.e. near $E = E_0$.

Now; scattering amplitude (Partial wave analysis)

$$F(\theta, \phi) = \frac{(2l+1)}{k} e^{i\delta_l} \sin\delta_l P_l(\cos\theta)$$

So, l dependence:

$$F_l(\theta) = e^{i\delta_l} \sin\delta_l = \frac{\sin\delta_l}{e^{-i\delta_l}}$$

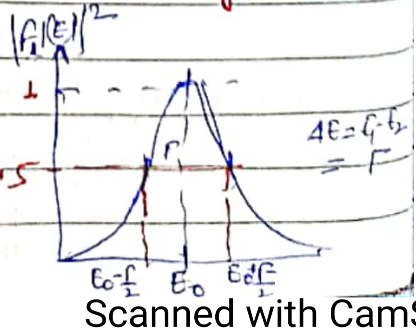
$$F_l(E) = \frac{\sin\delta_l(E)}{\cos\delta_l(E) - i\sin\delta_l(E)}$$

$$F_l(E) = \frac{1}{-\frac{2}{\Gamma}(E - E_0) - i} = \frac{-\Gamma/2}{(E - E_0) + i\frac{\Gamma}{2}}$$

$$|F_l(E)|^2 = F_l(E) F_l^*(E) = \frac{-\frac{\Gamma}{2}}{(E-E_0) + i\Gamma/2} \times \frac{-\frac{\Gamma}{2}}{(E-E_0) - i\Gamma/2}$$

$$|F_l(E)|^2 = \frac{\Gamma^2/4}{(E - E_0)^2 + \frac{\Gamma^2}{4}}$$

Breit Wigner formula



At half point $|F_l(E)|^2 = \frac{1}{2}$

then; $\dfrac{\Gamma^2/4}{(E-E_0)+\Gamma^2/4} = \dfrac{1}{2}$

or; $(E-E_0)^2 + \Gamma^2/4 = 2\Gamma^2/4$

or; $(E-E_0)^2 = \dfrac{2\Gamma^2}{4} - \dfrac{\Gamma^2}{4} = \dfrac{\Gamma^2}{4}$

$E-E_0 = \dfrac{\Gamma}{2}$ $\quad$ or; $\boxed{E = E_0 \pm \dfrac{\Gamma}{2}}$

or; separation between half points is width of line $= \Gamma$

Total cross section:

$$\sigma_T = \frac{4\pi}{k^2}(2l+1)\,|f_l(E)|^2$$

$$\sigma_T = \frac{4\pi}{k^2}(2l+1)\frac{\Gamma^2/4}{(E-E_0)+\Gamma^2/4}$$

(Dispersion formula)

Breit Wigner Resonance formula:

It gives value of nuclear reaction cross-section in the nbd of a single resonance level formed by an incident particle with zero angular momentum and zero charge (spin and coulomb effect ignored)

~~The optical Model~~:

* for all cases, of bound or not ~~excited~~ compound nucleus, excitation by an incident particle can be treated as analogus to the osc. ~~The optical model — requires use of a complex potential. consider a travelling wave in potential V;~~ of electrical ckt by emw.

* In classical resonant ckt, energy is dissipated due to resistance; in nucleus damping or loss of energy arises due to possibility of decay.

* Nuclear states has finite life-time and hence a finite width Γ.

* A wavefunction of decaying state of mean energy E_0 may be written as:

$$\psi(r,t) = \psi(r)\, e^{-iE_0 t/\hbar}\, e^{-\Gamma t/2\hbar} \quad \text{— (1)}$$

→ corresponds to exponential decrease of intensity of excitation $|\psi(r,t)|^2$ with a time const τ $(=\hbar/\Gamma)$

⇒ decaying state is not a state of definite energy E ... $\{\psi(r)\, e^{-i(E/\hbar)t}\}$

$= \frac{h\omega}{2\pi}$

But it can be represented by superposition of states of slightly diff. energies E each with a different amp. $A(E)$

$$\psi(r,t) = \psi(r)\int_{-\infty}^{+\infty} A(E)\, e^{-iEt/\hbar}\, dE \qquad (2)$$

using fourier analysis, the energy E can be grouped about a mean E_0 with a spread of the order of $\Gamma = \hbar/\lambda$.

we get;

$$e^{-\Gamma t/2\hbar} = e^{-\lambda t/2} = \int_{-\infty}^{\infty} A(E)\, e^{-i(E-E_0)t/\hbar}\, dE. \qquad (3)$$

As per fourier theorem, $f(t)$ can be representated:

$$f(t) = \frac{1}{2\pi}\lim_{\Omega\to\infty}\int_{-\Omega}^{\Omega} e^{-i\omega t}\, d\omega \int_{-\infty}^{\infty} e^{i\omega t'} f(t')\, dt' \qquad (4)$$

Applying this to eq (3), we get.

$$e^{-\lambda t/2} = \frac{1}{2\pi}\lim_{\Omega\to\infty}\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega \int_{\infty} e^{i\omega t'} e^{-\lambda t'/2}\, dt' \qquad (4')$$

comparing (4) & (4'), we get.

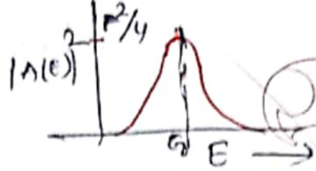
$$A(E) = \frac{1}{2\pi\hbar}\int_0^{\infty} e^{[i(E-E_0)/\hbar - \lambda/2]t'}\, dt'$$

$$\Rightarrow A(E) = \frac{1}{2\pi}\,\frac{1}{(E-E_0)+i\hbar\lambda/2}$$

→ Assume decaying system is prepared at $t=0$

Now, probability of finding the system in energy E is —

$$|A(E)|^2 = \frac{1}{4\pi^2}\,\frac{1}{(E-E_0)^2 + \left(\frac{\hbar\lambda}{2}\right)^2} \simeq \frac{1}{4\pi^2}\,\frac{1}{(E-E_0)^2+\frac{\Gamma^2}{4}}$$



$$|A(E)|^2 = \frac{1}{4\pi^2} \frac{1}{(E-E_0)^2 + \Gamma^2/4} \quad -(5)$$

$|A(E)|^2$ — max — $E = E_0$

$|A(E)|^2 \rightarrow$ decrease. $E > E_0$, $E < E_0$

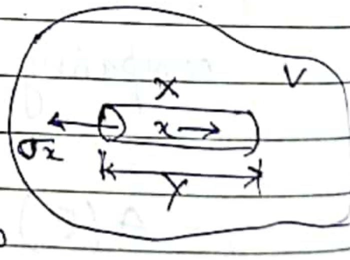
gives – level shape; exactly similar to pure radiative decay except that particle emission is now included by using the total width Γ instead of the radiative width Γ_r.

Cross Section for excitation of the levels by collision of particle x with nucleus X is therefore,

$$\sigma_x = \frac{C}{(E-E_0)^2 + \Gamma^2/4} \quad -(6)$$

C — is a constant; to determine C let us consider:

V — volume of nucleus. contains X & x



State – quantized — no. of states of motion of particles with momentum between p & p+dp

$= (4\pi p^2 dp V)/h^3$

The probability of formation of the compound nucleus per unit time =

No of possible states of motion × probability that nucleus X is in small volume $\sigma_x v$ swept out by effective collision ...

$E = \frac{p^2}{2m}$ $E' \cdot 2m \cdot \frac{m\,dE}{p} = \frac{mu}{4\pi\sqrt{E}}$

$dE = \frac{4\pi p^2\,dp}{4\pi m}\cdot \frac{v^2 m}{}$

$$dP = \frac{4\pi p^2 dp}{h^3} V \cdot \frac{\sigma_x v}{V} \quad \text{— (7)}$$

Substituting σ_x from (6), and integrating over the energy spectrum —

$$P = \int_{-\infty}^{\infty} dP = \int_{-\infty}^{\infty} \frac{4\pi}{h^3} v \cdot p^2 \frac{C}{(E-E_0)^2 + \Gamma^2/4}\, dp$$

$$= \int_{-\infty}^{\infty} \frac{4\pi C}{h^3} \frac{1}{(E-E_0)^2 + \frac{\Gamma^2}{4}}\, dE$$

We assume, variation of the channel wavelength 'λ' of the particle over the level width Γ may be neglected, then probability of formation/sec.

$$= \frac{4\pi C}{h\lambda^2} \cdot \frac{2}{\Gamma} \cdot \left[\tan^{-1} \frac{2(E-E_0)}{\Gamma}\right]_{-\infty}^{\infty}$$

$$= \frac{4\pi C}{h\lambda^2} \cdot \frac{2}{\Gamma} \cdot \pi = \frac{C}{\hbar \pi \lambda\!\!\!^{-2} \Gamma} \quad \text{— (8)}$$

[probability of decay per sec $= \Gamma_x/\hbar$]

Γ_x — ~~probability~~ Partial width of the compound nucleus.

In equilibrium — probability of formation per sec. would equal to the probability of decay per sec. of the excited state back into the system — $X + x$. Hence :

$$C/\pi\hbar\lambda\!\!\!^{-2}\Gamma = \Gamma_x/\hbar$$

$$\text{or;} \quad \boxed{C = \pi \lambda\!\!\!^{-2} \Gamma \Gamma_x} \quad \text{— (9)}$$

Substituting this value in eq. (6) the cross section for formation of the levels becomes:

$$\sigma_x(E) = \frac{\pi \lambda^2 \Gamma \Gamma_x}{(E-E_0)^2 + \Gamma^2/4} \quad \text{——} (10)$$

For the process $X(x,y)Y$, we obtain reaction cross-section as:

$$\sigma_{x} = \sigma_{xy}(E) = \frac{\sigma_x \Gamma_y}{\Gamma} \quad \text{——} (11)$$

$$= \frac{\pi \lambda^2 \Gamma_x \Gamma_y}{(E-E_0)^2 + \Gamma^2/4} \quad \text{——} (12)$$

Here, Γ_x, Γ_y are the partial widths defined as:

$$\Gamma_x \tau_x = \hbar \quad \& \quad \Gamma_y \tau_y = \hbar.$$

where, τ_x and τ_y are (partial level width) mean life times for (x, y).

Now;

If spin is considered, the right hand side, must be multiplied by the factor

$$g(c) = \frac{2I_c+1}{(2I_x+1)(2I_X+1)} \quad \text{——} (13)$$

I_x — total ang. momentum of particle x

I_c — " — " — of compd nucl.

I_X — " — of target X

Such that: $$\left.\begin{aligned} I_c &= I_x + I_x + I_x \\ &\& \quad \pi_x \pi_x (-1)^{l_x} = \pi_c \end{aligned}\right\} \quad (13)$$

Hence, eq (12) can be written as:

$$\boxed{\sigma_r = \pi \bar{\lambda}^2 \frac{(2I_c+1)}{(2I_x+1)(2I_x+1)} \frac{\Gamma_x \Gamma_y}{(E-E_0)^2 + \Gamma^2/4}} \quad (14)$$

This is the Breit Wigner resonance formula.

This is in particular for (n,γ), is:

$$\sigma(n,\gamma) = \pi \bar{\lambda}^2 \frac{(2I_c+1)}{2(2I_x+1)} \frac{\Gamma_x \Gamma_y}{(E-E_0)^2 + \Gamma^2/4}.$$

Max. at $\boxed{E = E_0}$ & is equal to

$$\sigma^{max}_{(n,\gamma)} = 4\pi \bar{\lambda}^2 g(c) \Gamma_n \Gamma_x / \Gamma^2.$$

for $E = E_0 \pm \Gamma/2$ $\quad \sigma = \frac{1}{2}\sigma_{max}$ — Γ_{FWHM}.

A sharp resonance corresponds to narrow width Γ and an excited state of a long life.