Numerical Solution of ordinary differential equations

by

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A first order **Initial Value Problem** (IVP) is defined as a first order differential equation together with specified initial condition at $x = x_0$:

$$\frac{dy}{dx} = f(x, y) \qquad \text{with} \quad y(x_0) = y_0$$

There exist several methods for finding solutions of differential equations.

The classical methods for approximate solution of an IVP are:

i) Picard Iteration method

ii) Taylor Series method

Picard Iteration Method:

Picard method is an iterative method. An iterative method gives a sequence of approximations $y_1(x)$, $y_2(x)$, …,$y_k(x)$,…to the solution of differential equations such that the nth approximation is obtained from one or more prevoius approximations.

PICARD'S METHOD

Consider the first order equation

$$dy/dx = f(x, y) \qquad ...(1)$$

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad ...(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign. *As a first approximation* y_1 *to the solution*, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

Similarly, *a third approximation is* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx.$

Continuing this process, a sequence of functions of x, *i.e.*, y_1, y_2, y_3 ... is obtained each giving a better approximation of the desired solution than the preceding one.

$$y_{k+1}(x) = y_0 + \int_{x_0}^{x} f(x, y_k)\,dx \; ; \quad k = 0, 1, 2, \ldots$$

<u>Assignments</u>

Q . Using Picard's process of successive approximation, obtain a solution upto fifth approximation of the equation dy/dx = y + x, such that y = 1 when x = 0.

Solution. (*a*) We have $y = 1 + \int_0^x (y + x)\, dx.$

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\, dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

Q . Find the value of y for x = 0.1 by Picard's method, given that

$\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0) = 1.

Solution. We have $y = 1 + \int_0^x \frac{y - x}{y + x}\, dx$

First approximation. Put $y = 1$ in the integrand, giving

$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left|-x + 2 \log(1 + x)\right|_0^x = 1 - x + 2 \log(1 + x)$$

Second approximation. Put $y = 1 - x + 2 \log(1 + x)$ in the integrand, giving

$$y_2 = 1 + \int_0^x \frac{1 - x + 2\log(1 + x) - x}{1 - x + 2\log(1 + x) + x}\, dx = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ we obtain

$$y(0.1) = 1 - (.1) + 2 \log 1.1 = 0.9828.$$

Problems

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0$, $y_0 = 1$ upto third approximation. *(Mumbai, 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem : $y' = x^2/(y^2 + 1)$, $y(0) = 0$. *(Marathwada, 2008)*
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and $y = 1$ at $x = 0$, using Picard's method.