Identical Particles in Quantum Mechanics

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Identical Particles

In quantum mechanics, a (quantum) particle is described by a wave packet of finite size. The simultaneous exact specification of position (spread of wave packet) and momentum of the particle is restricted by the Heisenberg's uncertainty principle.

Therefore, there is no way to keep track of individual particles separately, especially if they interact with each-other to an appreciable extent.

In quantum mechanics, the wave functions of the particles overlap considerably and hence the quantum particles are indistinguishable.

Physical Meaning of Identity

Identical particles in a system remain unaltered by interchanging them.

In quantum mechanics, identical particles can be substituted for each-other with no change in physical situation of the system.

However, with the spin consideration, identical particles can be distinguished, if they have different spin components along some particular axis e.g. z-axis, which remain unchanged during elastic collisions.

Symmetric and Antisymmetric Wave Functions

Schrodinger equation for n identical particles is

$$H\,(1,2,\text{---}n)\ \psi\,(1,2,\text{---}n,t) = i\hbar\ \partial/\partial t\ \psi\,(1,2,\text{---}n,t) \qquad \text{------} \quad (1)$$

where each number represents all coordinates (position and spin) of one of the particles.

Hamiltonian H is symmetrical in its arguments due to identity of particles, which means the particles can be substituted for each-other without changing the Hamiltonian H or any other observable.

There are two kinds of solutions of wave function ψ of eq. (1) that have symmetric properties of particular interest.

(i) **Symmetric wave function ψ_S** : A wave function is symmetric, if the interchange of any pair of particles among its arguments leave the wave function unchanged.

(ii) **Antisymmetric wave function ψ_A** : A wave function is antisymmetric, if the interchange of any pair of particles among its arguments changes the sign of the wave function.

Symmetric character of a wave function does not change with time i.e. if ψ_S (*or* ψ_A) is symmetric (*or antisymmetric*) at a particular time t, then $H\psi_S$ (*or* $H\psi_A$) and hence $\partial\psi_S/\partial t$ (*or* $\partial\psi_A/\partial t$) and the integration of wave function ψ_S (*or* ψ_A) are always symmetric (*or antisymmetric*).

If P is an exchange operator, then

$$P\,\psi_S(1,2) = \psi_S(2,1)$$

$$P\,\psi_A(1,2) = -\,\psi_A(2,1)$$

Construction of symmetric and antisymmetric wave functions from unsymmetrized functions : Exchange Degeneracy

If the arguments of wave function ψ are permuted in any way, then the resulting wave function is also a solution of equation (1). n! Solutions can be obtained from any one solution, each of which corresponds to one of the n! permutation of the n arguments of ψ. Any linear combination of these functions is also a solution of the wave equation (1). The sum of all these functions is symmetric (unnormalized) wave function ψ_S ,since interchange of any pair of particles changes any one of the component function into another of them and the latter into the former, leaving the entire wave function unchanged.

An antisymmetric unnormalized wave function can be constructed by adding all the permuted wave functions that arise from original solution by means of an even number of interchanges of pairs of particles (cyclic ones) and subtracting the sum of all the permuted wave functions that arise by means of an odd number of interchanges of pairs of particles in the original solution.

In cases where Hamiltonian does not depend on time, stationary state solutions

$$\psi(1,2,\text{---}n,t) = \phi(1,2,\text{---}n)\, e^{-iEnt/\hbar} \qquad \text{-----} \quad (2)$$

can be found and the time independent Schrodinger's eqn. can be written as

$$H(1,2,\text{---}n)\ \phi(1,2,\text{---}n) = E\,\phi(1,2,\text{---}n) \qquad \text{-----} \quad (3)$$

There are n! solutions of this equation (eigen functions) derived from ϕ (1,2,---n) by means of permutations of its arguments belonging to the same eigen value E. Any linear combination of these eigen functions is also an eigen function [solution of eq.(3)] belonging to eigen value E. Hence, the system is degenerate and this type of degeneracy is called exchange degeneracy.

For a system of two particles, Schrodinger time independent wave equation is

$$H(1,2)\,\psi(1,2) = E\,\psi(1,2) \qquad \text{-----} \quad (4)$$

2! = 2 solutions of this equation are ψ (1,2) and ψ (2,1) and correspond to a single energy state E.

Symmetric wave function can be written as

$$\psi_S = \psi(1,2) + \psi(2,1)$$

and antisymmetric wave function can be written as

$$\psi_A = \psi(1,2) - \psi(2,1)$$

For a system of three particles, Schrodinger time independent wave equation is

$$H(1,2,3)\,\psi(1,2,3) = E\,\psi(1,2,3)$$

This equation has following 3! = 6 solutions corresponding to the same eigen value E :

$$\psi(1,2,3),\ \psi(2,3,1),\ \psi(3,1,2),\ \psi(1,3,2),\ \psi(2,1,3),\ \psi(3,2,1)$$

Out of these six functions, those arising by an even number of interchanges of the pairs of particles in original wave function ψ (1,2,3) are :

$$\psi(1,2,3),\ \psi(2,3,1),\ \psi(3,1,2)$$

and the functions arising by an odd number of interchanges of pair of particles in original wave function ψ (1,2,3) are :

$$\psi(1,3,2),\ \psi(2,1,3),\ \psi(3,2,1)$$

Therefore, symmetric (unnormalized) wave function can be written as :

$$\psi_S = \{\psi(1,2,3) + \psi(2,3,1) + \psi(3,1,2)\} + \{\psi(1,3,2) + \psi(2,1,3) + \psi(3,2,1)\}$$

and antisymmetric (unnormalized) wave function as :

$$\psi_A = \{\psi(1,2,3) + \psi(2,3,1) + \psi(3,1,2)\} - \{\psi(1,3,2) + \psi(2,1,3) + \psi(3,2,1)\}$$

Particle Exchange Operator

Particle exchange operator $P_{1,2}$ is defined as :

$$P_{1,2}\ \psi(r_1 s_1; r_2 s_2) = \psi(r_2 s_2; r_1 s_1)$$

If the two particles are identical, then the Hamiltonian must be invariant under interchange of particles i.e. energy of the system remains the same, if we merely relabel the particles.

Eigen values and Eigen functions of Particle Exchange Operator

The eigen value equation for the particle exchange operator is :

$$P_{12}\ \psi(1,2) = \alpha\ \psi(1,2)$$

where α is the eigen value of operator $P_{1,2}$ in state ψ (1,2).

Operating again,

$$P_{12}^2\ \psi(1,2) = P_{12}\ P_{12}\ \psi(1,2) = P_{12}\ \alpha\ \psi(1,2) = \alpha\ P_{12}\ \psi(1,2) = \alpha^2\ \psi(1,2)$$

From the definition of particle exchange operator, we have

$$P_{12}\ \psi(1,2) = \psi(2,1)$$

Operating again,

$$P_{12}^2\ \psi(1,2) = P_{12}\ \psi(2,1)$$

i.e.

$$P_{12}^2\ \psi(1,2) = \psi(1,2)$$

Therefore,

$$\alpha^2 = 1 \quad \text{or} \quad \alpha = \pm 1$$

i.e. eigen value of particle exchange operator are ± 1.

Eigen functions of particle exchange operator corresponding to eigen values +1 and -1 are symmetric and antisymmetric.

$$P_{12}\ \psi_S = \psi_S \qquad \text{and} \qquad P_{12}\ \psi_A = -\psi_A$$

This may be seen as follows :

$$\psi_S = \psi(1,2) + \psi(2,1)$$

$$P_{12}\psi_S = P_{12}[\psi(1,2) + \psi(2,1)] = \psi(2,1) + \psi(1,2) = \psi_S$$

Also,

$$\psi_A = \psi(1,2) - \psi(2,1)$$

$$P_{12}\ \psi_A = P_{12}[\psi(1,2) - \psi(2,1)] = \psi(2,1) - \psi(1,2) = -\psi_A$$

Thus, particle exchange operator applied twice brings the particles back to their original configuration and hence produces no change in the wave function.

Commutation relation of Particle Exchange Operator with Hamiltonian

We have,

$$P_{12}\,\psi\,(1,2) = \psi\,(2,1)$$

$$P_{12}\,H(1,2)\,\psi\,(1,2) = H(2,1)\,\psi\,(2,1) = H(1,2)\,\psi(2,1) = H(1,2)\,P_{12}\psi(1,2)$$

[since Hamiltonian H is symmetric i.e. H(1,2) = H(2,1)]

$$[P_{12}\,H(1,2) - H(1,2)\,P_{12}]\,\psi\,(1,2) = 0$$

As ψ(1,2) is non-zero,

$$P_{12}\,H(1,2) - H(1,2)\,P_{12} = 0$$

$$[P_{12},H] = 0$$

Thus, particle exchange operator commutes with Hamiltonian.

Distinguishablility of Identical Particles

Two identical particles are distinguishable if the sum of probability density of individual wave functions of the two states is equal to the probability density associated with the symmetrised wave functions i.e.

$$|\psi\,(1,2)|^2 + |\psi\,(2,1)|^2 = |\{\psi\,(1,2) \pm \psi\,(2,1)\}|^2 = |\psi\,(1,2)|^2 + |\psi\,(2,1)|^2 \pm 2\,\mathrm{Re}\,[\psi\,(1,2)\,\psi^*(2,1)]$$

Thus, if the space and spin co-ordinates of the exchange degenarate functions (of the two particles) are different, the interference term i.e. $2\,\mathrm{Re}\,[\psi\,(1,2)\,\psi^*(2,1)] = 0$ and particle wave functions do not overlap, *making the particles distinguishable*.

Pauli's Exclusion Principle

For a system of non-interacting n identical particles, the approximate (unperturbed) Hamiltonian of the system is equal to the sum of Hamiltonian function for the separate particles i.e.

$$H_0(1,2,\text{------}n) = H_0(1) + H_0(2) + \text{------} + H_0(n)$$

and the approximate energy eigen function is a product of one particle eigen function of H_0.

$$\psi(1,2,\text{---}n) = \phi_a(1)\,\phi_b(2)\text{-----}\,\phi_k(n)$$

with $E = E_a + E_b + \text{-------} + E_k$.

$$H_0(1)\,\phi_a(1) = E_a\,\phi_a(1)\text{, etc.}$$

If the particles are Fermions (electrons), then for a system of two non-interacting particles, an antisymmetric wave function can be written as a determinant

$$\phi_A(1,2) = \frac{1}{\sqrt{2}}\,[\phi_a(1)\,\phi_b(2) - \phi_b(1)\,\phi_a(2)]$$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix}\varphi a(1) & \varphi a(2)\\ \varphi b(1) & \varphi b(2)\end{vmatrix}$$

For a system of n non-interacting Fermi particles, the antisymmetric energy wave function can be written as

$$\phi_A(1,2,\text{-----}n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \phi_a(1) & \phi_a(2) & \text{---} & \phi_a(n) \\ \phi_b(1) & \phi_b(2) & \text{---} & \phi_b(n) \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \phi_k(1) & \phi_k(2) & \text{---} & \phi_k(n) \end{vmatrix}$$

This is called 'Slater determinant'.

Since a determinant vanishes if any two rows are identical, it is obvious that ϕ_A will vanish if more than one particle is in the same state i.e. if a = b.

This is Pauli's exclusion principle which states that no two particles described by antisymmetric wave functions can exist in the same quantum state.

Connection between Spin and Statistics

The symmetry property of wave function has close relationship with spin of the particle.

(i) The identical particles having integral spin quantum numbers are described by symmetric wave functions i.e.

$$P\,\psi_S\,(1,2,3,\text{----}r,\text{----}s,\text{----}n) = +\ \psi_S\,(1,2,3,\text{----}s,\text{----}r,\text{----}n)$$

Such particles obey Bose-Einstein statistics and are called Bosons e.g. photons (spin 1) and neutral He-atoms in normal state (spin 0).

(ii) The identical particles having half odd integral spin quantum numbers are described by antisymmetric wave functions i.e.

$$P\,\psi_A\,(1,2,3,\text{----}r,\text{----}s,\text{----}n) = -\ \psi_A\,(1,2,3,\text{----}s,\text{----}r,\text{----}n)$$

Such particles obey Fermi-Dirac statistics and are called Fermions e.g. electrons, protons, neutron, muons (spin 1/2).

Spin Angular Momentum

Spin is intrinsic angular momentum (a quantum concept with no classical analogue).

It is independent of r, θ and ϕ.

It has two intrinsic states i.e. two z-components of spin momentum.

Electron has intrinsic angular momentum characterized by quantum number ½.

Intrinsic electron spin is a vector **S** (spin quantum number = ½) with s_z = +1/2 and -1/2 and the respective spin wave functions are α and β (α and β are orthogonal).

Spin angular momentum of electron :

$$S\,\alpha = \sqrt{s(s+1)}\hbar\,\alpha = \sqrt{3}/2\,\hbar\,\alpha \qquad s_z\,\alpha = m_s\,\hbar\,\alpha = \tfrac{1}{2}\,\hbar\,\alpha$$

$$S\,\beta = \sqrt{s(s+1)}\hbar\,\beta = \sqrt{3}/2\,\hbar\,\beta \qquad s_z\,\beta = m_s\,\hbar\,\beta = -\tfrac{1}{2}\,\hbar\,\beta$$

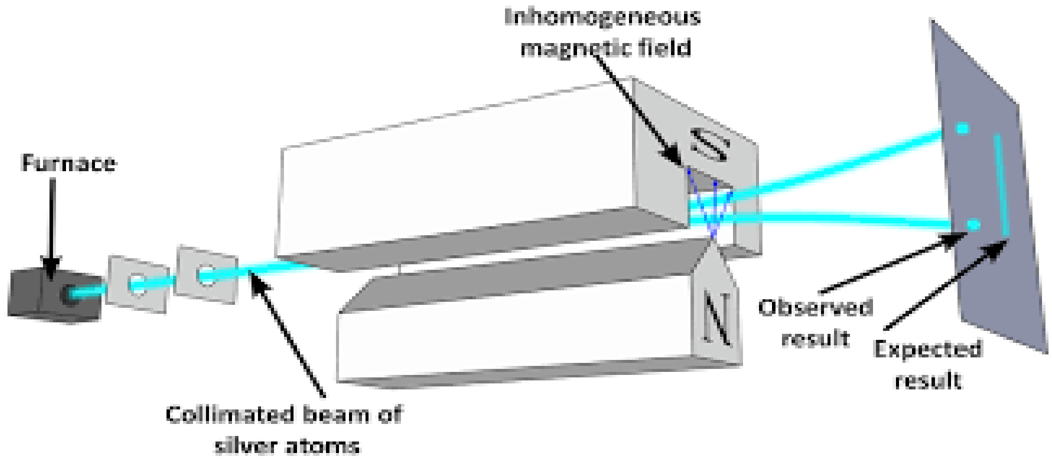
$$\int \alpha^*\beta\, d\sigma = \int \beta^*\alpha\, d\sigma = 0 \qquad \int \alpha^*\alpha\, d\sigma = \int \beta^*\beta\, d\sigma = 1$$

[σ is spin variable.]

Stern-Gerlach Experiment

In 1922, at the University of Frankfurt (Germany), Otto Stern and Walther Gerlach, did fundamental experiments in which beams of silver atoms were sent through inhomogeneous magnetic fields to observe their deflection. These experiments demonstrated that these atoms have quantized magnetic moments that can take two values.

Inhomogeneous magnetic field was generated with –ve field gradient in z-direction i.e. $\partial B/\partial z < 0$. The magnetic field is strong near N-pole and weak near S-pole, as in fig. When vapour of silver-beam was passed through this inhomogeneous B-field, it was observed to split into two traces, which were attributed to the two spin state of m_z.



Explanation :

Force acting on Ag-atom is

$$\mathbf{F} = - \text{grad } U = \text{grad } \mathbf{m.B} \qquad (U = - \mathbf{m.B})$$

$$F = m \cos\theta\, \partial B/\partial z \qquad\qquad a_z = F/M_0 = m/M_0 \cos\theta\, \partial B/\partial z\,, \quad t = L/v$$

$$z = ½\, a_z t^2 = ½\, m \cos\theta\, (L^2/M_0 v^2)\, \partial B/\partial z$$

Classically, cos θ can have all possible values from -1 to +1, giving smear of Ag-beam after passing through B-field (not observed in this experiment).

But quantum mechanically, due to space quantization, cos θ = ± 1. So,

$$z = \pm ½\, m\, (L^2/M_0 v^2)\, \partial B/\partial z$$

Goudsmit and Uhlenbeck hypothesis

(i) Each electron has spin angular momentum **S**, whose component in z-direction can have values $s_z = \pm ½$.

(ii) Each electron has spin magnetic moment $\boldsymbol{\mu}_s = - (e/m_0 c)\, \mathbf{S}$.

Spin obeys commutation relations :

$$[S_x,S_y] = i\hbar\varepsilon_{jkl}\, S_l$$

where ε_{jkl} is Levi-Civita symbol. It follows that S^2 and S_z are :

$$S^2|s,m_s\rangle = \hbar^2 s(s+1)|s,m_s\rangle$$

$$S_z|s,m_s\rangle = \hbar m_s|s,m_s\rangle$$

Spin raising and lowering operators acting on these eigen vectors give :

$$S_{\pm}|s,m_s\rangle = \hbar\sqrt{[s(s+1) - m_s(m_s\pm 1)]}\ |s,m_s\rangle$$

where $S_{\pm} = S_x \pm iS_y$

All quantum mechanical particles possess an intrinsic spin, which is quantized (though this value may be zero, too), such that the state function of the particle is $\psi(\mathbf{r},\boldsymbol{\sigma})$; where σ is out of the following discrete set of values

$$\sigma \in \{-s\hbar, -(s-1)\hbar, --- 0 -- +(s+1)\hbar, +s\hbar\}$$

Bosons have integer spin and fermions have half-integer spin. Total angular momentum is the sum of orbital angular momentum and the spin.

Pauli matrices

Quantum mechanical operators associated with spin ½ observables are :

$$\hat{S} = (\hbar/2)\ \sigma$$

where in Cartesian components :

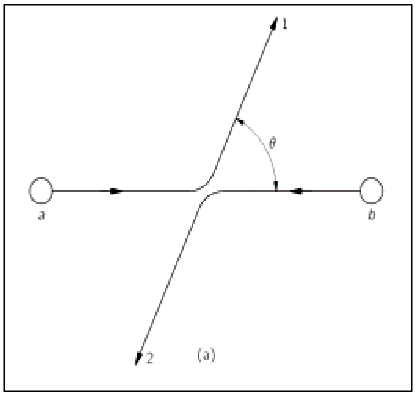
$$S_x = (\hbar/2)\ \sigma_x\ ,\ S_y = (\hbar/2)\ \sigma_y\ ,\ S_z = (\hbar/2)\ \sigma_z\ ;$$

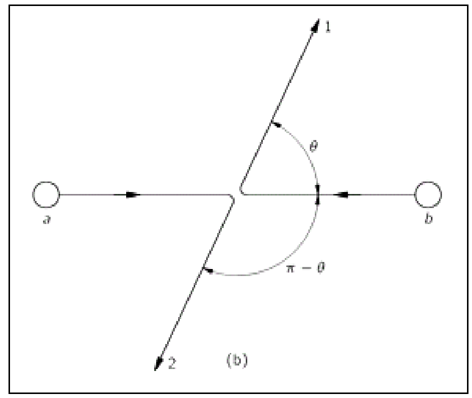
σ_x , σ_y and σ_z are Pauli's spin matrices.

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Collision of Identical Particles

Let us consider the collision of two particles a and b, in which particle 'a' scatters in direction 1 and b in direction 2, as shown in fig-(a). If f(θ) be the amplitude for this process, then the probability P_1 of observing such an event is proportional to $|f(\theta)|^2$.





(Scattering of identical particles in CM system)

If, as a result of collision, particle 'a' enters into counter 2 and 'b' into counter 1, then the probability P_2 for this process is proportional to $|f(\pi-\theta)|^2$.

If 'a' and 'b' are identical particles, then the two processes shown in fig-a and b cannot be distinguished. The amplitude that either 'a' or 'b' goes into counter 1, while the other goes into counter 2, is the sum of the amplitudes for the two processes.

An interchange of two identical particles [$1 \leftrightarrows 2$ i.e. $\mathbf{r} \rightarrow (-\mathbf{r})$ in CM frame] does not affect the position vector of centre of mass, which is ½ ($\mathbf{r}_1 + \mathbf{r}_2$), but changes the sign of relative position vector $\mathbf{r}$ (= $\mathbf{r}_1 - \mathbf{r}_2$).

The asymptotic form of unsymmetrized scattering wave function in the centre of mass co-ordinate system is given by

$$u(\mathbf{r}) \xrightarrow[r\to\infty]{} e^{ikz} + r^{-1} f(\theta,\phi) e^{ikr}$$

where r, θ, ϕ are the polar co-ordinates of the relative position vector **r**.

Since the polar co-ordinate of the vector **– r** are r, π - θ, ϕ + π, we have

$$u(-\mathbf{r}) \rightarrow e^{-ikz} + r^{-1} f(\pi - \theta, \phi + \pi)\, e^{ikr}$$

The asymptotic forms of the symmetric and antisymmetric wave functions are given by

$$(e^{ikz} \pm e^{-ikz}) + [f(\theta,\phi) \pm f(\pi - \theta, \phi + \pi)] r^{-1} e^{ikr}$$

with upper and lower signs respectively.

Hence, differential cross section in the centre of mass co-ordinate system is the square of the magnitude of the bracket term

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2 + |f(\pi - \theta, \phi + \pi)|^2 \pm 2\,\mathrm{Re}\,[f(\theta,\phi) f^*(\pi - \theta, \phi + \pi)]$$

Taking into account the effect of spin on the collision of two identical particles, we have for bosons, which have symmetric wave functions :

$$\psi_{sym} = \phi_s \chi_s \qquad \text{or} \quad \psi_{sym} = \phi_A \chi_A$$

[here ϕ is space wave function & χ is spin wave function.]

Fermions have anti symmetric wave functions : $\psi_{antisym} = \phi_s \chi_A$ or $\psi_{antisym} = \phi_A \chi_s$

If spin wave function $\chi_s = |sm\rangle$ is symmetric, then space wave function ϕ is anti-symmetric (so that ψ is anti-symmetric) and

$$d\sigma_A/d\Omega = |f(\theta) - f(\pi-\theta)|^2$$

and if spin wave function $\chi_A = |sm\rangle$ is anti-symmetric, then space wave function ϕ is symmetric (so that ψ is anti-symmetric) and

$$d\sigma_s/d\Omega = |f(\theta) + f(\pi-\theta)|^2$$

Since symmetric spin wave function $\chi_s \Rightarrow |sm\rangle = |1,0\rangle, |1,\pm1\rangle \rightarrow 3$

and anti-symmetric spin wave function $\chi_A \Rightarrow |sm\rangle = |0,0\rangle \rightarrow 1$

$$(d\sigma/d\Omega)_{fermions} = \tfrac{3}{4}\, d\sigma_A/d\Omega + \tfrac{1}{4}\, d\sigma_s/d\Omega$$

$$= \tfrac{3}{4} |f(\theta) - f(\pi-\theta)|^2 + \tfrac{1}{4} |f(\theta) + f(\pi-\theta)|^2$$

$$= \tfrac{3}{4} [f(\theta) - f(\pi-\theta)] [f(\theta) - f(\pi-\theta)]^* + \tfrac{1}{4} [f(\theta) + f(\pi-\theta)] [f(\theta) + f(\pi-\theta)]^*$$

$= \frac{3}{4} [|f(\theta)|^2+|f(\pi-\theta)|^2 - 2Re \{f (\theta)f^*(\pi-\theta)\}]+ \frac{1}{4} [|f(\theta)|^2 +|f(\pi-\theta)|^2 + 2Re \{f(\theta)f^*(\pi-\theta)\}]$

$= |f(\theta)|^2 + |f(\pi-\theta)|^2 + [- \frac{3}{4} x2Re \{f(\theta) f^*(\pi-\theta)\}] + \frac{1}{4} x2Re \{f(\theta) f^*(\pi-\theta)\}]$

$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - Re \{f(\theta) f^*(\pi-\theta)\}$

The result for 2s = odd (for fermions) or even (for bosons) can be summarized by writing the scattering cross-section σ (θ) as

$$\sigma(\theta) = |f(\theta)|^2 + |f(\pi - \theta)|^2 + \frac{(-1)^{2s}}{2s+1} 2 \, \mathrm{Re}\, [f(\theta)f^*(\pi - \theta)]$$

At $\theta = \pi/2$:

$$(d\sigma/d\Omega)_{fermions} = |f(\pi/2)|^2 + |f(\pi/2)|^2 - Re \{f^*(\pi/2) f(\pi/2)\} = 2|f(\pi/2)|^2 - |f(\pi/2)|^2$$
$$= |f(\pi/2)|^2$$

But $(d\sigma/d\Omega)_{classical} = |f(\theta)|^2 + |f(\pi-\theta)|^2$

(In classical mechanics, there is no interference term "Re $f(\theta) f^*(\pi-\theta)$".)

At $\theta = \pi/2$:

$$(d\sigma/d\Omega)_{classical} = 2|f(\pi/2)|^2$$

Therefore, at $\theta = \pi/2$: $(d\sigma/d\Omega)_{classical} = 2(d\sigma/d\Omega)_{fermions}$

i.e. quantum differential scattering cross-section is half of classical differential scattering cross-section.

If the space wave function is symmetric, then differential scattering cross-section

$$d\sigma_s/d\Omega = |f(\theta) + f(\pi-\theta)|^2$$

If the space wave function is anti-symmetric, then differential scattering cross-section

$$d\sigma_A/d\Omega = |f(\theta) - f(\pi-\theta)|^2$$

Electron Spin Functions

In Pauli's theory, eigen values of the operator σ_x is

$$\sigma_x\Psi = \alpha\Psi \quad \text{-------------------} \quad (1)$$

where α is the eigen value of σ_x in state Ψ.

Since, $\Psi = \begin{bmatrix}\Psi 1\\ \Psi 2\end{bmatrix}$ and $\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$; equation (1) takes the form

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Psi 1\\ \Psi 2\end{bmatrix} = \alpha\begin{bmatrix}\Psi 1\\ \Psi 2\end{bmatrix}$$

$$\begin{bmatrix}\Psi 2\\ \Psi 1\end{bmatrix} = \begin{bmatrix}\alpha\Psi 1\\ \alpha\Psi 2\end{bmatrix}$$

$$\Psi_2 = \alpha\Psi_1 \qquad \text{and} \qquad \Psi_1 = \alpha\Psi_2$$

Therefore, we have,

$$\alpha^2 = 1 \qquad \text{or} \quad \alpha = \pm 1$$

A convenient set of orthonormal one particle spin function is provided by the normalized eigen functions of L^2 and L_x matrices. In this case, the eigen functions are (2S+1) row and one column matrices.

For electron S =1/2, so eigen function matrices for electron have (2x1/2 +1) = 2-rows and 1-column.

The respective normalized wave functions for x-component of the wave function are

$$\Psi_x(1/2) = 1/\sqrt{2}\begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad \Psi_x(-1/2) = 1/\sqrt{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

In a similar way,

$$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\Psi 1\\ \Psi 2\end{bmatrix} = \beta\begin{bmatrix}\Psi 1\\ \Psi 2\end{bmatrix}$$

$$\begin{bmatrix}-i\Psi 2\\ i\Psi 1\end{bmatrix} = \begin{bmatrix}\beta\Psi 1\\ \beta\Psi 2\end{bmatrix}$$

$$-i\Psi_2 = \beta\Psi_1 \quad \text{and} \quad i\Psi_1 = \beta\Psi_2$$

Therefore, we have,

$$\beta^2 = 1 \quad \text{or} \quad \beta = \pm 1$$

So, the normalized wave functions for y-component of the wave function are

$$\Psi_y(1/2) = 1/\sqrt{2}\begin{bmatrix}1\\ i\end{bmatrix} \quad \text{and} \quad \Psi_y(-1/2) = 1/\sqrt{2}\begin{bmatrix}1\\ -i\end{bmatrix}$$

Similarly, the normalized wave functions for z-component of the wave function are

$$\Psi_z(1/2) = 1/\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad \Psi_z(-1/2) = 1/\sqrt{2}\begin{bmatrix}0\\1\end{bmatrix}$$

Spin Functions for n-particle system / Electron spin function for many electron systems

Let us consider n identical particles 1,2,----,n. The spin wave function of a single particle is completely determined by the specification of (2s+1) numbers, whereas the space wave function involves the specification of a continuously infinite set of numbers (which is equivalent to a continuous function of the space co-ordinates).

A convenient set of orthonormal one-particle spin functions is given by the normalized eigen functions of total angular momentum J^2 and its component J_z matrices. Then eigen functions are (2s+1)-row, one-column matrices that have zeros in all positions except one. For example, if s = 3/2, the four spin eigen functions are :

$$\Psi(3/2) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \Psi(1/2) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \Psi(-1/2) = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \Psi(-3/2) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

and correspond to S_z eigen values 3/2 ħ, ½ ħ, -1/2 ħ, -3/2 ħ.

The orthonormality is demonstrated by multiplying the Hermitian adjoint of one spin function into itself or another function

$$[0 \quad 1 \quad 0 \quad 0]\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 1, \qquad [0 \quad 1 \quad 0 \quad 0]\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = 0, \text{ ----- etc.}$$

Hermitian adjoint of Ψ(1/2) and Ψ(1/2) **Hermitian adjoint of Ψ(1/2) and Ψ(-1/2)**

Symmetric or antisymmetric many-particle wave functions can be constructed from unsymmetrized solutions that include the spin.

It is sometimes convenient to choose the unsymmetrized solutions to eigen functions of the square of the magnitude of the total spin of the identical particles $(\mathbf{S_1} + \mathbf{S_2} + \text{------} + \mathbf{S_n})^2$ and of the z-component of this total spin $(S_{1z} + S_{2z} + \text{------} + S_{nz})$.

These quantities are constants of motion, if the Hamiltonian does not contain interaction terms between the spins and other angular momenta.

In addition, such functions are often useful as zero-order wave functions when the spin interactions are weak enough to be regarded as a perturbation.

There is no loss of generality in choosing the unsymmetrized solutions in this way, since any solutions can be expressed as a linear combination of total spin eigen functions.

Effect of identity and spin

The interaction between identical particles does not depend on spin. In order to take into account the identity and spin of the two electrons, we need form an antisymmetric wave function from the products of $\chi_i^{+}(\mathbf{r_1}, \mathbf{r_2})$ and appropriate spin functions. The spin functions can be taken to be the set of the following four symmetrised combinations :

(+ +)

(1)

$1/\sqrt{2}$ [(+ -) + (- +)]

(- -)

$1/\sqrt{2}$ [(+ -) - (- +)]

where (+) = σ_x(-) and (-) = σ_x(+).

In the elastic scattering of an electron from a hydrogen atom (which may be considered as core of infinite mass, as compared to electron), the spin of incident electron does not have any definite relation to the spin of atomic electrons.

We can use either of these sets of spin functions, calculate the scattering with each of the four spin states of a set and then average the results with equal weights for each state.

The first three of the spin functions (1) are symmetric and must be multiplied by the antisymmetric space function $\chi_i^+(\mathbf{r_1}, \mathbf{r_2}) - \chi_i^+(\mathbf{r_2}, \mathbf{r_1})$; the fourth spin function is antisymmetric and must be multiplied by $\chi_i^+(\mathbf{r_1}, \mathbf{r_2}) + \chi_i^+(\mathbf{r_2}, \mathbf{r_1})$.

The asymptotic forms of the symmetrised space functions for large values of one of the electron co-ordinates, say $\mathbf{r_1}$ are

$$\chi_i^+(\mathbf{r_1}, \mathbf{r_2}) \pm \chi_i^+(\mathbf{r_2}, \mathbf{r_1}) \rightarrow C\,[\exp(i\mathbf{k_\alpha}.\mathbf{r_1}) + r^{-1} e^{ik\alpha r}\,[f_D(\theta) \pm f_E(\theta)]\,\omega_\alpha(r_2) \qquad \text{--- (2)}$$

where ω_α = core bound initial wave function, f_D = direct or non-exchange elastic scattering amplitude for which incident electron is scattered and atomic electron is left in its original state; f_E = exchange elastic scattering amplitude.

The dots represent atomic excitation and θ is the angle between $\mathbf{r_1}$ and $\mathbf{k_\alpha}$.

The differential cross-section is computed with the upper sign in one quarter of the collisions and with lower sign in three quarters of the cases.

Thus, we obtain

$$\sigma(\theta) = \tfrac{1}{4}\,|f_D(\theta) + f_E(\theta)|^2 + \tfrac{3}{4}\,|f_D(\theta) - f_E(\theta)|^2 \qquad \text{--- (3)}$$

Equation (3) may also be derived without explicit reference to spin wave functions by making use of the fact that the particles having different spin components are distinguishable.

If in half the collisions, the electrons have different sum of direct and exchange cross-sections i.e. $|f_D(\theta)|^2 + |f_E(\theta)|^2$ and in the other half where the electrons are indistinguishable, the antisymmetric space function is used.

Thus, we obtain

$$\sigma(\theta) = \tfrac{1}{2}\,\{|f_D(\theta)|^2 + |f_E(\theta)|^2\} + \tfrac{1}{2}\,|f_D(\theta) - f_E(\theta)|^2 \qquad \text{--- (4)}$$

Obviously, equation (4) is same as equation (3).

Thus, in the classical limit, where the identical particles are distinguishable, the interference term $2\text{Re}[f(\theta,\phi)f^*(\pi - \theta, \phi + \pi)] = 0$ and the scattering cross-section $\sigma(\theta, \phi)$ becomes just the sum of differential cross-section for observation of the incident particle $(|f(\theta, \phi)|^2$ and $(|f(\pi - \theta, \phi + \pi)|^2$.

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2 + |f(\pi - \theta,\, \phi + \pi)|^2$$

If f is independent of ϕ, then the scattering per unit solid angle will be symmetric about $\theta = 90^0$ in the centre of mass co-ordinate system.

References used :

1. *Quantum Mechanics by* ***L. I. Schiff****,*
2. *Quantum Mechanics – Concepts & Applications by* **Nouredine Zettili** &
3. *Quantum Mechanics by* ***V. K. Thankappan****.*

Assignments

1. What is the physical meaning of identity ?

2. How symmetric and antisymmetric wave functions can be constructed from unsymmetrized functions ?

3. Discuss distinguishability of identical particles. Explain Pauli's exclusion principle with the help of Slater determinant.

4. What is spin angular momentum ? Describe Stern-Gerlac experiment. Write Goudsmit and Uhlenbech hypothesis.

5. Obtain expression for scattering cross-section for the collision of two identical particles.