

EIGEN FUNCTIONS & EIGEN VALUES of Angular Momentum

$$[\hat{L}^2, \hat{L}_z] = 0$$

The solution of rigid rotor is $\psi(\phi, \theta) = \Theta_{l,m}(\theta) \cdot \Phi_m(\phi) = Y_{l,m}(\theta, \phi)$

Spherical Harmonics

The Hamiltonian of rigid rotor & its eigenvalues:-

$$\hat{H} = \frac{\hat{L}^2}{2I} \quad \text{and} \quad E = \frac{l(l+1) \hbar^2}{8\pi^2 I} \times 2\pi$$

$$\hat{L}^2 = -\frac{\hbar^2}{4\pi^2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$\hat{H} = -\frac{\hbar^2}{8\pi^2 I} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$\psi = \Theta(\theta) \Phi(\phi)$$

\downarrow

$$Y_{l,m}(\theta, \phi) \quad L^2 = l(l+1) \hbar^2$$

spherical harmonics.

$$\frac{\hbar}{2\pi} = \hbar$$

Thus the eigenfunctions of angular momentum operator \hat{L}^2 is same as that of rigid rotor

$$\frac{\hbar}{2\pi} = \hbar = 1$$

\therefore The eigenvalue equation is given by

$$\hat{A}\psi = (\alpha)\psi$$

$$\hat{L}^2 Y_{l,\pm m}(\theta, \phi) = l(l+1) \frac{\hbar^2}{4\pi^2} Y_{l,\pm m}(\theta, \phi) = l(l+1) \hbar^2 Y_{l,m}(\theta, \phi)$$

$$= l(l+1) Y_{l,m}(\theta, \phi)$$

$$\hat{L}^2 Y_{l,m}(\theta, \phi) = l(l+1) \hbar^2 Y_{l,m}(\theta, \phi)$$

$[\hat{L}^2, \hat{L}_z] = 0$ \therefore The eigen function of \hat{L}_z can be chosen as same i.e, $Y_{l,\pm m}(\theta, \phi)$

where $\Theta_{l,m}(\theta) = \sqrt{\frac{2l+1}{2}} \frac{(l-|m|)!}{(|l|-|m|)!} P_l^{|m|}(\cos\theta)$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(\pm im\phi)$$

Eigen-values of \hat{L}_z :

$$\hat{\Delta}\psi = \alpha \cdot \psi$$

$$\hat{L}_z\psi = \lambda\psi$$

$$\hat{L}_z = \frac{h}{2\pi i} \frac{\partial}{\partial \phi}$$

$$\frac{h}{2\pi i} \frac{\partial \psi}{\partial \phi} = \lambda \psi$$

$$\frac{\partial \psi}{\partial \phi} = \frac{2\pi i}{h} \lambda \cdot \psi$$

$$e^{ix} = \cos x + i \sin x$$

$$\frac{1}{\psi} d\psi = \frac{2\pi i}{h} \lambda \cdot d\phi$$

$$\textcircled{+} = 0 \text{ to } 2\pi$$

$$\phi + 2\pi$$

$$\psi = \exp \left[\frac{2\pi i}{h} \lambda \phi \right] = \cos \left(\frac{2\pi \lambda}{h} \phi \right) + i \sin \left(\frac{2\pi \lambda}{h} \phi \right)$$

$\psi \Rightarrow$ single valued

$$\exp \left[\frac{2\pi i}{h} \lambda \phi \right] = \exp \left[\frac{2\pi i}{h} \lambda (\phi + 2\pi) \right]$$

$$\exp [im\phi] = \exp [im(\phi + 2\pi)]$$

where $m = \frac{2\pi}{h} \lambda$

$$= \exp(im\phi) \cdot \exp(i2\pi m)$$

$$\exp(2\pi i m) = 1 \quad \text{or}$$

$$\cos(2\pi m) + i \sin(2\pi m) = 1$$

Possible only if $m = 0, \pm 1, \pm 2, \dots, \pm n$

$$\therefore \lambda = m \frac{h}{2\pi} \quad m = \frac{2\pi}{h} \lambda$$

λ , eigen value of L_z must be zero or integral multiple of $h/2\pi$

Thus eigen value equation for L_z can be written as :

$$\hat{L}_z Y_{l,m}(\theta, \phi) = \pm m \frac{h}{2\pi} Y_{l,m}(\theta, \phi) = m h Y_{l,m}(\theta, \phi) = m Y_{l,m}(\theta, \phi)$$