

COMMUTATION RELATIONS
of Angular momentum operators.

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x$$

$$\hat{L}_x \hat{L}_y = \frac{\hbar}{i} \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right] \cdot \frac{\hbar}{i} \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right]$$

$$= \frac{\hbar^2}{i^2} \left[y \frac{\partial}{\partial z} \left(z \frac{\partial}{\partial x} \right) - y \frac{\partial}{\partial z} \left(x \frac{\partial}{\partial z} \right) - z \frac{\partial}{\partial y} \left(z \frac{\partial}{\partial x} \right) + z \frac{\partial}{\partial y} \left(x \frac{\partial}{\partial z} \right) \right]$$

$$= -\hbar^2 \left[yz \frac{\partial^2}{\partial z \partial x} + y \frac{\partial}{\partial x} - yx \frac{\partial^2}{\partial z^2} - z^2 \frac{\partial^2}{\partial y \partial x} + zx \frac{\partial^2}{\partial y \partial z} \right]$$

$$i^2 = -1 ; \quad \frac{\partial z}{\partial z} = 1 ; \quad \frac{\partial x}{\partial z} = 0 ; \quad \frac{\partial z}{\partial y} = 0 ; \quad \frac{\partial x}{\partial y} = 0$$

$$\frac{h}{2\pi} = \hbar$$

$$\hat{L}_x = \frac{\hbar}{i} \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right]$$

$$\hat{L}_y = \frac{\hbar}{i} \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right]$$

$$\begin{aligned}
\hat{L}_y \hat{L}_x &= \frac{\hbar}{i} \left[z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right] \cdot \frac{\hbar}{i} \left[y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right] \\
&= -\hbar^2 \left[z \frac{\partial}{\partial x} \left(y \frac{\partial}{\partial z} \right) - z \frac{\partial}{\partial x} \left(z \frac{\partial}{\partial y} \right) - x \frac{\partial}{\partial z} \left(y \frac{\partial}{\partial z} \right) + x \frac{\partial}{\partial z} \left(z \frac{\partial}{\partial y} \right) \right] \\
&= -\hbar^2 \left[z y \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial x \partial y} - z \frac{\partial^2}{\partial y \partial z} - x y \frac{\partial^2}{\partial z^2} + x z \frac{\partial^2}{\partial z \partial y} \right]
\end{aligned}$$

$$\begin{aligned}
\hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x &= -\hbar^2 \left[y z \frac{\partial^2}{\partial z \partial x} + y \frac{\partial}{\partial x} - y x \frac{\partial^2}{\partial z^2} - z^2 \frac{\partial^2}{\partial y \partial x} + z x \frac{\partial^2}{\partial y \partial z} \right] \\
&\quad + \hbar^2 \left[z y \frac{\partial^2}{\partial x \partial z} - z^2 \frac{\partial^2}{\partial x \partial y} - z \frac{\partial^2}{\partial y \partial z} - x y \frac{\partial^2}{\partial z^2} + x z \frac{\partial^2}{\partial z \partial y} \right]
\end{aligned}$$

$$\begin{aligned}\hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x &= -\hbar^2 \left[y \frac{\partial}{\partial z} - x \frac{\partial}{\partial y} \right] \\ &= \hbar^2 \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]\end{aligned}$$

$$\hat{L}_z = \frac{\hbar}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right]$$

$$\begin{aligned}[\hat{L}_x, \hat{L}_y] &= \hat{L}_x \hat{L}_y - \hat{L}_y \hat{L}_x \\ &= i\hbar \hat{L}_z\end{aligned}$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$

\hat{L}_x, \hat{L}_y & \hat{L}_z

Do not commute with each other, so simultaneously cannot be specified

By convention, \hat{L}_z is chosen

$$\begin{aligned}
 [\hat{L}^2, \hat{L}_z] &= [(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2), \hat{L}_z] \\
 &= (\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)\hat{L}_z - \hat{L}_z(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2) \\
 &= \hat{L}_x^2\hat{L}_z + \hat{L}_y^2\hat{L}_z + \hat{L}_z^2\hat{L}_z - \hat{L}_z\hat{L}_x^2 - \hat{L}_z\hat{L}_y^2 - \hat{L}_z\hat{L}_z^2 \\
 &= (\hat{L}_x^2\hat{L}_z - \hat{L}_z\hat{L}_x^2) + (\hat{L}_y^2\hat{L}_z - \hat{L}_z\hat{L}_y^2) + (\hat{L}_z^2\hat{L}_z - \hat{L}_z\hat{L}_z^2) \\
 &= [\hat{L}_x, \hat{L}_z] + [\hat{L}_y, \hat{L}_z] + [\hat{L}_z, \hat{L}_z]
 \end{aligned}$$

$$\begin{aligned}
 [\hat{L}_x, \hat{L}_z] &= -i\hbar[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x] && \text{Total angular momentum} \\
 [\hat{L}_y, \hat{L}_z] &= i\hbar[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x] && \downarrow \text{commutes with all components.} \\
 [\hat{L}_z, \hat{L}_z] &= 0
 \end{aligned}$$

$\therefore [\hat{L}^2, \hat{L}_z] = 0 ; [\hat{L}^2, \hat{L}_x] = 0 ; [\hat{L}^2, \hat{L}_y] = 0$
 $\hat{L}^2 \& \text{ any one component} \Rightarrow \text{specify simultaneously}$