Thermodynamic Behavior of an Ideal Bose-Gas

An assembly of bosons is termed as Bose-Einstein gas, where Bosons are identical, indistinguishable elementary particles of zero or integral spin.

Example of Bosons particles are photons, helium atoms at low temperature.

As the gas is assumed to be perfect (ideal), the interaction betⁿ its particles is negligible so that energy may be regarded as entirely translation in character.

We consider a perfect BE gas of N bosons

Let these particles be distributed among quantum groups or states are $n_1, n_2, \ldots n_i$

where approximate constant energies are $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_i$ respectively

From the Grand Canonical partition function

$$Z_G = \sum_{n=0}^{\infty} \exp(\beta\mu N) \sum_{n_i} e^{-\beta\varepsilon_i n_i}$$

$$Z_G = \sum_{n=0}^{\infty} \exp(\beta\mu N) Z_c$$

$$\boxed{\log Z_G = \frac{PV}{KT} = -\sum_i \ln(1 - z e^{-\beta\varepsilon_i})} \quad (1)$$

$P \rightarrow$ pressure

$V \rightarrow$ Volume

$T \rightarrow$ Temperature

$z \rightarrow$ fugacity

$z = \exp(\beta\mu)$

$z = \exp(\mu/KT)$

3

$$\boxed{N = \sum_i \langle n_i \rangle = \sum_i \frac{1}{z^{-1} e^{\beta \epsilon_i} - 1}} \quad (2)$$

We convert it into integral for this we need density of states

eq^n (1) and (2) can be written as

$$\boxed{\frac{P}{KT} = -\frac{2\pi}{h^3}(2m)^{3/2} \int_0^\infty \epsilon^{1/2} \ln(1 - z e^{-\beta\epsilon}) d\epsilon - \frac{1}{V}\ln(1-z)} \quad (3)$$

$$\boxed{\frac{N}{V} = \frac{2\pi}{h^3}(2m)^{3/2} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{z^{-1} e^{\beta\epsilon} - 1} + \frac{1}{V}\frac{z}{1-z}} \quad (4)$$

Density of states taken as for

3 Dimensional $\sim \epsilon^{1/2}$

2 Dimensional $\sim \epsilon^{0}$ (constant)

1 Dimensional $\sim \epsilon^{-1/2}$

$\epsilon = \frac{\hbar^2 k^2}{2m}$

Scanned with CamScanner

In eq^n 3 & 4 IInd term of eqn.

$z << 1$ $\qquad z \to [0, 1]$

This correspond to very large temperature

so the no. of particles is very low,
z is small No is very small

$$\boxed{N = N_0 + N_{ex}}$$

$N_0 \to$ no. of particles in ground state
$N_{ex} \to$ no. of particles in excited state

so the condensation is not possible.

If z values close to $\to 1$

then no. of particles No is increases
then eq 4 second term is not negligible
this can be written as

$$\boxed{\frac{z}{1-z} = N_0} \qquad N_0 = \frac{e^{\beta\mu}}{1 - e^{\beta\mu}}$$

$$\boxed{z = \frac{N_0}{N_0+1}} \qquad \boxed{\frac{1}{V}\ln(1-z) = \frac{1}{V}\ln(N_0+1)}$$

This term is negligible

Scanned with CamScanner

In eq^n 3 & 4

substituting $\boxed{\beta \epsilon = \frac{p^2}{2mkT} = x}$

$$\frac{P}{kT} = \frac{-2\pi(2mkT)^{3/2}}{h^3} \int_0^\infty x^{1/2} \ln(1 - z e^{-x})\, dx$$

$$\boxed{\frac{P}{kT} = \frac{1}{\lambda^3} g_{5/2}(z)} \quad — 6$$

$$\frac{N - N_0}{V} = \frac{2\pi(2mkT)^{3/2}}{h^3} \int_0^\infty \frac{x^{1/2} dx}{z^{-1} e^{x} - 1}$$

$$\boxed{\frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{3/2}(z)} \quad — 7$$

$$\boxed{\lambda = \frac{h}{(2\pi mkT)^{1/2}}} \quad — 8$$

$g_n(z)$ are B E function

$$g_n(z) = \frac{1}{\Gamma n}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^{x}-1}\,dx$$

$$= z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots$$

$$\boxed{g_{3/2}(z) = \zeta(3/2) = 2.612}$$

Reimann Zeta function

eq^n 7 can be written as

$$\frac{N - N_0}{V} \leq \left(\frac{2\pi m k T}{h^3}\right)^{3/2} \zeta(3/2)$$

$$\boxed{N_0 = N - \left\{ V \left(\frac{2\pi m k T}{h^3}\right)^{3/2} \zeta(3/2) \right\}}$$

9

BEC

$$\boxed{N > V T^{3/2} \frac{(2\pi m k)^{3/2}}{h^3} \zeta(3/2)} \quad (10)$$

if N and V constant and vary T

$$\boxed{T < T_C = \frac{h^2}{2\pi m k} \left\{ \frac{N}{V \zeta(\frac{3}{2})} \right\}^{2/3}} \quad (11)$$

$T_C \rightarrow$ a characteristic temperature that depends upon the particle mass m and the particle density N/V in the system.

i) $N_e \left\{ = N \left(\frac{T}{T_C}\right)^{3/2} \right\} \longrightarrow$ normal phase particle distributed in excited state $(\varepsilon \neq 0)$

ii) $N_0 \left\{ = (N - N_e) \right\} \longrightarrow$ a condensed phase particle accumulated in the ground state $(\varepsilon = 0)$

If $T \rightarrow T_c$ Then we have

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \approx \frac{3}{2}\,\frac{T_c - T}{T_c}} \quad (12)$$

$$\boxed{N_0 = N\left[1 - \frac{T}{T_c}\right]^{3/2}} \quad (13)$$

When the temperature of a B-E gas is lowered below the critical temperature T_c the number of particles in the ground state rapidly increases. This rapid increase in the population of the ground state for B-E gas is called the Bose Einstein condensation.

At the ground state $\varepsilon = 0$, particles do not contribute to the energy.

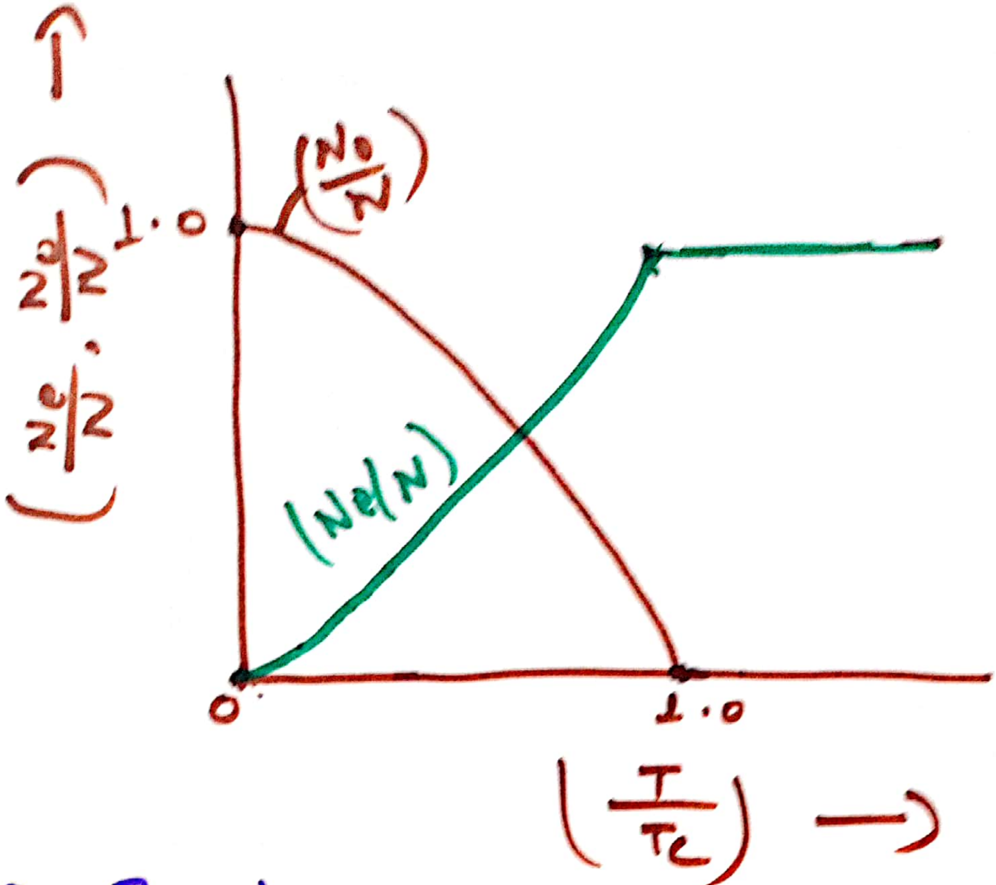

Fig ① Fraction of the normal phase and the condensed phase in an Ideal Bos Gas as a function of the temperature parameter (T/T_c)

For

He liquid T_c calculated → 3.12 K

but observed value → 2.19 K

which is close to calculated value.

The internal energy of the system is given by

$$= -\frac{\partial}{\partial \beta} \ln Z_G = KT^2 \left\{ \frac{\partial}{\partial T}\left(\frac{PV}{KT}\right) \right\}_V$$

$$\boxed{U = \frac{3}{2} KT \frac{V}{\lambda^3} g_{5/2}(z)} \quad — (14)$$

For NET

$U \propto T$

$$\boxed{\lambda \propto T^{-1/2}}$$

$U \propto T^{5/2}$

$U = A T^{5/2}$

$$\boxed{P = \frac{2}{3}\frac{U}{V}}$$

$P \propto T^{5/2}$

$$C_V = \frac{dU}{dT} = \frac{5}{2} A T^{3/2}$$

$$\boxed{C_V = \frac{5}{2}\frac{U}{T}}$$

$$\frac{C_V(T_c)}{NK} \simeq 1.925$$

$$\frac{C_V}{R} = \frac{3}{2}$$

$\frac{C_V}{k} = \frac{3}{2}$

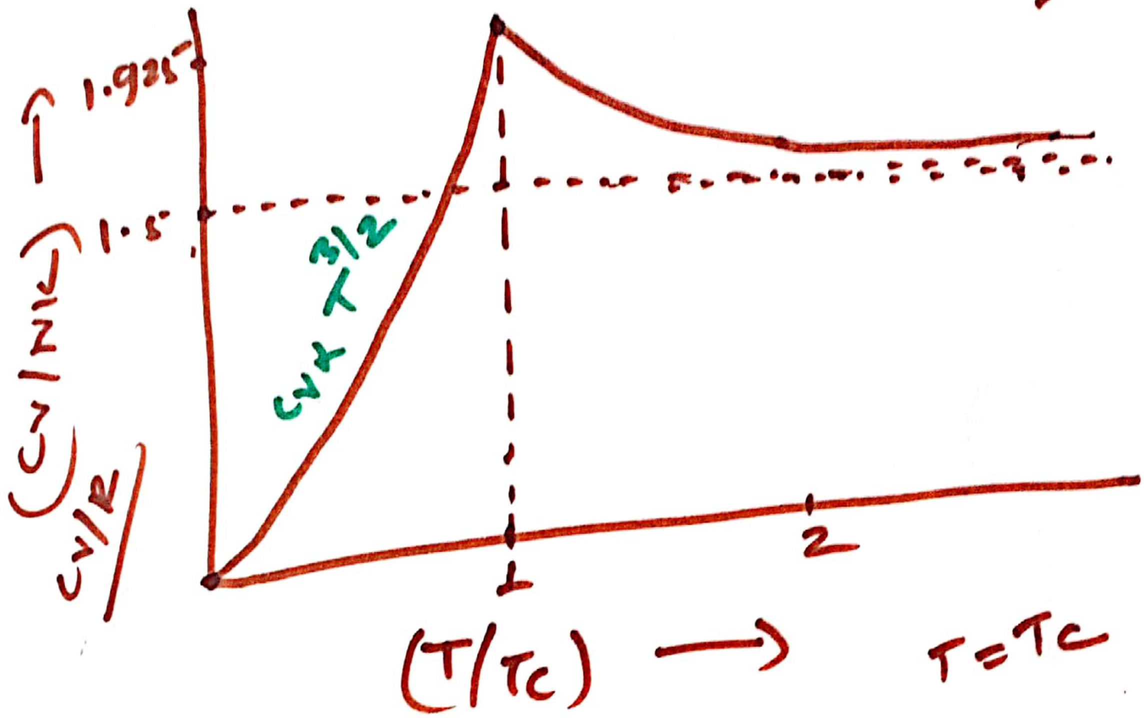

Fig ② The specific heat of an ideal Bose gas as a function of the temperature parameter (T/T_c).

References :-

) Statistical Physics by - Pathria

) Statistical Mechanic by - Saty Prakash