Radiation from an Accelerated charge at High velocity

Relativistic Generalisation of Larmour formula

Larmour's non-relativistic formula is

$$P = \frac{1}{4\pi\epsilon_0} \frac{2q^2a^2}{3c^3} \quad \text{--- (1)}$$

this formula can be generalised for arbitrary velocities of charge by using the fact that radiated em energy behaves under LT like the fourth component of a four vector

Since $P = dE_{rad}/dt$ $P \rightarrow$ Power

$dE_{rad} = P\,dt$

time t is also behaves under LT

i.e the Power P is lorentz invarient quantity if we can find a lorentz invarient w

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which reduces to non relativistic Larmour's formula (1) for $\beta << v/c$ $<<1$

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then we have the required relativistic generalization of Larmour formula

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of Larmour's formula, let us use the fact

$$a = \frac{F}{m} = \left[\frac{dp/dt}{m}\right] = \quad \ldots (2)$$

and write Larmor's non relativistic formula (1) in the form

$$P = \frac{1}{4\pi\epsilon_0} \frac{2q^2 (a \cdot a)}{3c^3}$$

substituting the value a ~~from~~ (2)

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{m^2c^3} \left(\frac{dp}{dt} \cdot \frac{dp}{dt}\right) \quad \ldots (2)$$

$m \rightarrow$ mass, $p \rightarrow$ momentum

for generalisation of eq^n (2)

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{m^2c^3} \left(\frac{dp_\mu}{d\tau} \; \frac{dp_\mu}{d\tau}\right) \quad \ldots (3)$$

$p_\mu \rightarrow$ charge particle's momentum energy four operator

given by

$$p_\mu = \left(p, \frac{iE}{c}\right) \quad ---- (4)$$

$d\tau \rightarrow$ Proper time interval, given by

$\beta = \frac{v}{c}$

$$dt = \frac{d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{d\tau}{\sqrt{1 - \beta^2}} = \gamma \, d\tau \quad \ldots (5)$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

now verifying eq^n (3) reduces to eq^n (1)

substituting as $\beta \rightarrow 0$

$$\frac{dP_\mu}{d\tau}\frac{dP_\mu}{d\tau} = \left(\frac{dP_1}{d\tau}\right)^2 + \left(\frac{dP_2}{d\tau}\right)^2 + \left(\frac{dP_3}{d\tau}\right)^2 + \ldots$$

we evaluate four vector scalar pro...

$$\left(\frac{dp}{d\tau}\right)^2 + \left\{\frac{d}{dt}\left(\frac{iE}{c}\right)\right\}^2 = \left(\frac{dp}{d\tau}\right)^2 - \frac{1}{c^2}\left(\ldots\right)$$

but $d\tau = dt/\gamma$

$$\frac{dP_\mu}{d\tau}\frac{dP_\mu}{d\tau} = \gamma^2\left(\frac{dp}{dt}\right) - \frac{\gamma^2}{c^2}\left(\frac{dE}{dt}\right)^2$$

substituting the value of $\frac{dP_\mu}{d\tau}$ in (3)

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2\gamma^2}{3m^2c^3}\left[\left(\frac{dp}{dt}\right) - \frac{1}{c^2}\left(\frac{dE}{dt}\right)^2\right] \ldots$$

Now reducing above eqn in terms of velocity and acceleration, let us u... relativistic formula

$$E = \frac{mc^2}{\sqrt{1-\beta^2}}, \quad p = \frac{mv}{\sqrt{1-\beta^2}} \quad \ldots (6)$$

so $$\frac{dE}{dt} = \frac{d}{dt}\left[mc^2(1-\beta^2)^{-1/2}\right]$$

$$= \frac{mc^2\,\vec{\beta}\cdot\vec{\beta}}{(1-\beta^2)^{3/2}}$$

$$= mc^2\gamma^3\,\vec{\beta}\cdot\dot{\vec{\beta}} \quad \ldots 8$$

and $$\frac{dp}{dt} = \frac{d}{dt}\left\{\frac{mv}{(\sqrt{1-\beta^2})}\right\} = \frac{d}{dt}\left\{\frac{mc\beta}{\sqrt{1-\beta^2}}\right\}$$

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$$\frac{dP}{dt'} = \vec{\dot\beta}$$

$$= \frac{mc(1-\beta^2) + \beta(\beta.\dot\beta)}{(1-\beta^2)^{3/2}}$$

$$= mc\gamma^3 \left[\dot\beta.(1-\beta)^2 + \beta(\beta.\dot\beta)\right] \quad \text{--- (9)}$$

Now subsituting the value of dE/dt and dP/dt from (8) + (9) in (6)

$$P = \frac{1}{4\pi\epsilon_0} \cdot \frac{2q^2\gamma^2}{3m^2c^3}\left[mc\gamma^3\left\{\dot\beta.(1-\beta)^2 + \beta(\beta.\dot\beta)\right\}. - \frac{1}{c^2} m^2c^2.c^2\gamma^3.\gamma^3\beta^2.\dot\beta^2\right]$$

(6) $$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}\gamma^6\left[(\dot\beta)^2 - \beta^2\dot\beta^2 + (\beta.\dot\beta)^2\right]$$

but $(\beta\times\dot\beta)^2 = \beta.\dot\beta - \beta^2\dot\beta^2$

$$\therefore \boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6}{c}\left[\dot\beta^2 - (\beta\times\dot\beta)\right]}$$

This is relavistic generalisation of Larmor's formula

Analysis of the formula

Case I If the motion of the particle is non relativistic

$\beta \to 0 \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \to 1$

$$\therefore P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}\left[(\vec{\dot\beta})^2\right]$$

but $\beta = a/c$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c}\frac{a^2}{c^2}$$

$$\boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}}$$

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Non relativistic Larmor's formula

Case II If the motion of particle is collinear (velocity u, acc a are parallel)

$$|\vec{\beta} \times \dot{\vec{\beta}}| = \beta\dot{\beta}\sin 0 = 0$$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6\dot{\beta}^2}{c} \qquad \dot{\beta} = \frac{a}{c}$$

$$\boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6 a^2}{c^3}}$$

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This type of radiation is Bremsstrahlung radiation, which is produced from accelerated electron with the assumption that the direction of motion of electrons does not change.

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Case III If the motion of the particle is such that its velocity + acceleration are perpendicular

$$|\beta \times \dot{\beta}| = \beta\dot{\beta}\sin 90 = \beta\dot{\beta}$$

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6}{c}\left(\dot{\beta}^2 - \beta^2\dot{\beta}^2\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^6}{c}\dot{\beta}^2(1-\beta^2)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^4}{c}\dot{\beta}^2$$

$$\boxed{P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2\gamma^4 a^2}{c^3}}$$

$$\dot{\beta} = \frac{a}{c}$$

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This type of radiation occurs in circular accelerator like synchroton or cyclotron, where energy loss is drastic