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Angular distribution of radiation emitted by an accelerated charge

Whenever the charge of particle moves, means → current → magnetic
→ magnetic field B

→ charge particle $\begin{cases} E \\ B \end{cases}$

Particle acceleration → change in velocity
∴ accelerated field depends on the velocity & acc and is given by

$$\bar{E}_a = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\bar{n}\times(\bar{n}-\bar{\beta})\times\dot{\bar{\beta}}}{(1-\bar{\beta}\cdot\bar{n})^3 R}\right] \quad \text{--- } 1(a)$$

But ~~Ma~~ magnetic field (induction) is

$$B_a = \frac{1}{c}(\bar{n}\times\bar{E}_a) \quad \text{--- } 1b$$

where $\beta = \bar{v}/c$, $\dot{\beta} = \text{Const.}$

$\bar{n} = R/R$ unit vector

Poynting vector $S = E\times H$

$H \rightarrow$ magnetic field intensity $= \frac{B_a}{\mu_0}$

$$\bar{S}_a = \bar{E}_a \times \bar{H}_a$$

$$= E_a \times \frac{B_a}{\mu_0} = \frac{1}{\mu_0}\bar{E}_a\times\bar{B}_a$$

Putting the value of B_a from 1(b)

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$$\bar{S}_a = \frac{\bar{E}_a \times \bar{n} \times \bar{E}_a}{\mu_0 c}$$

we know

$\Rightarrow (\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}$

$\Rightarrow \quad \bar{n}.\bar{n} = 1$

$$\therefore \bar{S}_a' = \frac{(\bar{E}_a . \bar{E}_a)\ \bar{n} - (\bar{E}_a . \bar{n})\bar{E}_a}{\mu_0 c}$$

but $\quad \bar{E}_a . \bar{n} = \bar{E}_a \cos 90^\circ = 0$

$$\Rightarrow \bar{S}_a = \frac{\bar{E}_a^2 . \bar{n}}{\mu_0 c} \quad \text{---- } 2a$$

The radial component of poynting vector is

$$\Rightarrow [\bar{S}_a . \bar{n}] = \frac{\bar{E}_a^2}{\mu_0 c} \quad \text{--- } 2b$$

$\Downarrow$

It shows that energy emitted per unit ~~time~~ area per unit Time detected at an observation point at time t' due to radiation emitted by the charge at time

$t' = t - \frac{R(t')}{c} \Rightarrow$ retarded time

where

$R = |\bar{X} - \bar{X}'|$

$t =$ real time

$X' =$ Position of source point

$X =$ " " observation point

So the energy radiated during a finite period

$$\Rightarrow W = \int_{t=T_1+\frac{R(t_1')}{c}}^{t=T_2+\frac{R(t_2')}{c}} (\bar{S_a}\cdot\bar{n})\,dt$$

substituting the value to t

$$\Rightarrow W = \int_{t'=T_1}^{t'=T_2} (\bar{S_a}\cdot\bar{n})\frac{dt}{dt'}\cdot dt'$$

$$\Downarrow$$

The term $(\bar{S_a}\cdot\bar{n})\frac{dt}{dt'}$ represents the power radiated per unit area

$\therefore$ The power radiated per unit solid angle

$$\Rightarrow \frac{dP(t')}{d\Omega} = \text{Power radiated per unit area} \times \frac{4\pi R^2}{4\pi}$$

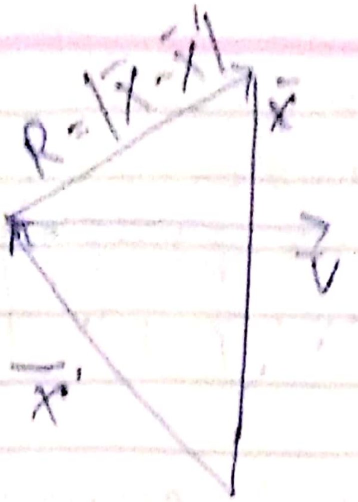
$$\frac{dP(t')}{d\Omega} = R^2(\bar{S_a}\cdot\bar{n})\frac{dt}{dt'} \quad \text{--- (3)}$$

but $t' = t - R/c$

$$\frac{dt'}{dt} = 1 - \frac{1}{c}\frac{dR}{dt'}\frac{dt'}{dt}$$

$$\left(1+\frac{1}{c}\frac{dR}{dt'}\right)\frac{dt'}{dt} = 1$$

$$\therefore \frac{dt'}{dt} = \frac{1}{1+\frac{1}{c}\frac{dR}{dt'}} \quad \text{--- 3(a)}$$



Charge at $\bar{x}'$ moving with velo. v

$x \to$ point of observation

$\therefore |x - x'| = R$ magnitude

But $\frac{dR}{dt'} = -\bar{V} \cdot \hat{n}$

now subsituting the value of dR/dt' in eqn 3(a)

$$\frac{dt'}{dt} = \frac{1}{1 + \frac{1}{c}(\bar{v} \cdot \bar{n})} = \frac{1}{1 - \left(\frac{\bar{V} - \bar{n}}{c}\right)}$$

$$= \frac{1}{1 - \bar{\beta} \cdot \bar{n}} \qquad \frac{\bar{V}}{c} = \bar{\beta}$$

$$\frac{dt}{dt'} = 1 - \bar{n}\bar{\beta} = K \quad \text{---} \; (4)$$

subsituting the value of $\frac{dt}{dt'}$ in (3) form (4)

$$\frac{dP(t')}{d\Omega} = KR^2(\bar{S_a} \cdot \bar{n}) \quad \text{---} (5)$$

$$= KR^2 \cancel{R^2} \frac{\bar{E_a}^2}{\mu_0 c} \qquad \text{from 2b}$$

now using eqn 1b where the value of $\bar{E_a}$ 1a

$$\frac{dP(t')}{d\Omega} = \frac{KR^2}{\mu_0 c}\left(\frac{q}{4\pi\epsilon_0 c}\right)^2 \left[\frac{\bar{n} \times \bar{n} \times \bar{\beta}) \times \dot{\bar{\beta}}}{(1 - \bar{\beta} \cdot \bar{n})^3 R}\right]^2$$

Putting $k = 1 - \bar{n}\beta$.

and $\mu_0 \epsilon_0 = \frac{1}{c^2}$

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \left[\frac{[\bar{n} \times (\bar{n} - \bar{\beta}) \times \dot{\bar{\beta}}]^2}{(1 - \bar{\beta} \cdot \bar{n})^5} \right] \quad —(6)$$

This is the general expression for the angular distribution of power radiated by an accelerated charge

Conditions - Case I

1 When velocity + acceleration are parallel

$$\bar{\beta} \times \dot{\bar{\beta}} = 0$$

$\therefore$ eqn (6) becomes

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \left[\frac{\bar{n} \times (\bar{n} \times \dot{\bar{\beta}})}{(1 - \bar{n} \cdot \bar{\beta})^5} \right]^2$$

using properties of double cross product

$\Rightarrow \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \cdot b - (\bar{a} \cdot \bar{b}) \cdot c$

and $\bar{n} \cdot \bar{n} = 1$

$$\Rightarrow \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \left[\frac{(\bar{n} \cdot \dot{\bar{\beta}})\bar{n} - (\bar{n} \cdot n) \cdot \dot{\bar{\beta}}}{(1 - \bar{n}\bar{\beta})^5} \right]^2$$

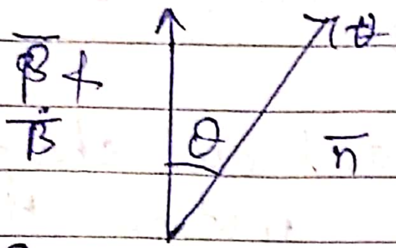
$$\Rightarrow \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \left[\frac{(\bar{n} - \dot{\bar{\beta}}) \cdot \bar{n} - \dot{\bar{\beta}}}{(1 - \bar{n}\beta)^5} \right]^2$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0} \frac{q^2}{4\pi c} \frac{|\dot{\bar{\beta}}|^2 - (\bar{n} \cdot \dot{\bar{\beta}})^2}{(1 - \bar{n}\bar{\beta})^5}$$

squaring above terms, we get

$$\Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c}\frac{\left[(\bar{n}\cdot\dot{\bar\beta})^2 - 2(\bar{n}\cdot\dot{\bar\beta})^2\right] + \dot{\bar\beta}^2}{(1-\bar{n}\bar\beta)^5}$$

$$\Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\bar\beta}^2 - (\bar{n}\cdot\dot{\bar\beta})^2}{(1-\bar{n}\bar\beta)^5} \quad \text{— (7)}$$



from fig. $\bar{n}\cdot\dot{\bar\beta} = \dot{\bar\beta}\cos\theta$

$\bar{n}\cdot\bar\beta = \bar\beta\cos\theta$

so eq^n (7) becomes

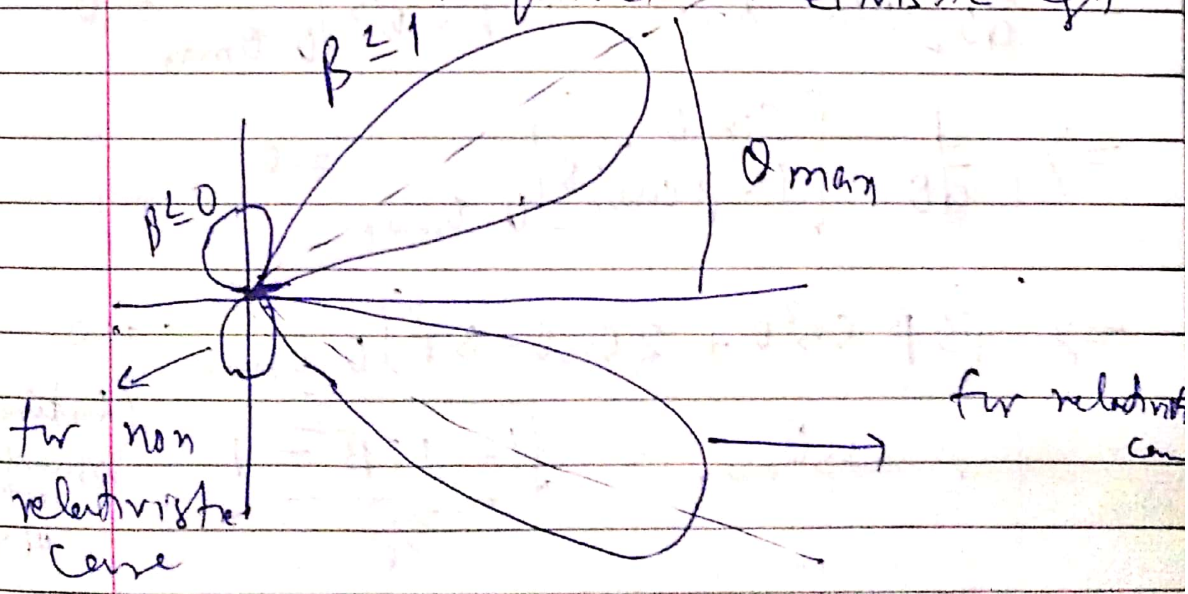
$$\Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q^2}{4\pi c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5}$$

but $\dot{\bar\beta} = \frac{\bar{a}}{c}$

$$\Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q^2\bar{a}^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta^2\cos\theta)^5} \quad \text{— (8)}$$

$\Downarrow$

This is required relativistic eq^n



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For non relativistic case
$\vec{\beta} << 1 \quad \therefore \vec{\beta}' \to 0$

So eq^n (8) reduces to

$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{q^2 a^2}{4\pi c^3} \sin^2\theta \quad \text{—— (9)}$$

Thus for an accelerated charge in non relativistic motion the angular distribution shows a simple $\sin^2\theta$ behaviour

but as $\vec{\beta}' \to 1$, we get θ_{max} (where intensity is max)

The value of θ_{max}, calculated by the condition

$$\frac{d}{d\theta}\left(\frac{dP}{d\Omega}\right) = 0 \quad \text{for} \quad \theta = \theta_{max}$$

using eq^n (8)

$$\Rightarrow \frac{d}{d\theta}\left[\frac{1}{4\pi\epsilon_0} \frac{q^2 a^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{\theta=\theta_{max}} = 0$$

$$\Rightarrow \frac{d}{d\theta}\left[\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right]_{\theta=\theta_{max}} = 0$$

$$\Rightarrow \left[3\beta\cos^2\theta + 2\cos\theta - 5\beta\right]_{\theta=\theta_{max}} = 0$$

$$\therefore \cos\theta_{max} = \frac{\sqrt{1+15\beta^2} - 1}{3\beta}$$

(taking square root of eq^n)

$$\Rightarrow \theta_{max} = \cos^{-1}\frac{\sqrt{1+15\bar{\beta}^2}-1}{3\bar{\beta}} \quad \text{——— 10a}$$

taking limit $\bar{\beta} \to 1$

$$\Rightarrow \theta_{max} = \frac{1}{2}\sqrt{1-\bar{\beta}^2} = \frac{1}{2\gamma} \quad \text{--- 10b}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

θ_{max} is very small for relativistic particles, so angular distribution is confined to very narrow cone in the direction of motion.

The total power radiated can be obtained by integrating eqn (8)

$$\Rightarrow P = \int \frac{dP}{d\Omega} d\Omega$$

$$\Rightarrow P = \frac{1}{4\pi\epsilon_0}\frac{q^2\bar{a}^2}{4\pi c^3}\int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} d\Omega$$

choosing the direction of $\vec{\beta}$ or $\dot{\vec{\beta}}$ along z-axis

$$\Rightarrow P = \frac{1}{4\pi\epsilon_0}\frac{q^2\bar{a}^2}{4\pi c^3}\int_0^{\pi} \frac{\sin^2\theta}{(1-\bar{\beta}\cos\theta)^5} d\theta \int_0^{2\pi} d\phi$$

$$\Rightarrow P = \frac{1}{4\pi\epsilon_0}\frac{q^2\bar{a}^2}{4\pi c^3}\frac{4}{3(1-\bar{\beta}^2)^3} \cdot 2\pi$$

$$\Rightarrow P = \frac{1}{4\pi\epsilon_0}\frac{q^2\bar{a}^2}{4\pi c^3}\frac{8\pi}{3(1-\bar{\beta}^2)^3}$$

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$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2 \dot{a}^2 \gamma^6}{c^3}$$

This is final eqn of case 1 and
I exactly same result as eqn of
acc. charge particle at high velocity
where its velocity + acc. are parallel

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