Eigen value + eigen vectors

Let $A = [a_{ij}]_{n \times n}$ be a given matrix of order $n \times n$

and let $AX = \lambda X$ —— ①

$X = 0$ soln of ① for all λ

thus a value of λ for which ① & $X \neq 0$ is called eigen value and non zero solution of X is called eigen vector corresponding to that eigen value.

eqn ① can also be written as

$$AX = \lambda I X$$

$$(A - \lambda I) X = 0 \quad —— ②$$

$I \rightarrow$ unit matrix of order $n \times n$

Eqn ② $\Rightarrow$ n homogeneous linear equation in n unknown

$$x_1(a_{11} - \lambda) + a_{12} x_2 + \cdots + a_{1n} x_n = 0$$

$$a_{21} x_1 + (a_{22} - \lambda) x_2 + \cdots + a_{2n} x_n = 0 \quad —— ③$$

$$a_{n1} x_1 + a_{n2} x_2 + \cdots + (a_{nn} - \lambda) x_n = 0$$

$\Rightarrow$ These are system of homogeneous equation and will have non zero solution if the coefficient determinant vanishes

$$|A - \lambda I| = 0$$

Eq.-(4) Characteristic equation

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{2n} \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{bmatrix} = 0 \quad \text{--- (4)}$$

on solving above eqn, we get an eqn of degree n in λ, which gives n root, each root is called eigen value or latent root and corresponding to each eigen value we can obtain non zero vector $X \neq 0$

$$X = [x_1 \; x_2 \; x_3 \; \cdots \; x_n]$$

This non zero vector is called eigen vector

1. Find eigen value and eigen vector of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol. The char. eqn of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 + \cancel{\lambda^2} - 2\lambda - 5\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda-1)(\lambda-6) = 0$$

$$\lambda = 1, 6$$

$\therefore$ eigen value of A are 1, 6

Eigen-vector corresponding to $\lambda = 6$

Let $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ be an eigen vector corresponding to $\lambda = 6$.

$$(A - 6I)X = 0$$

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + 4y = 0$$
$$x - 4y = 0$$

We obtain only one independent eqn

$$-x + 4y = 0$$

$$\frac{x}{4} = \frac{y}{1}$$

$\therefore X = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as the required eigen vector for $\lambda = 6$

Eigen vector corresponding to $\lambda = 1$

$$(A - I)X = 0$$

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\therefore \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x + 4y = 0$

$x + y = 0$

we obtain only one independent equation $x + y = 0$

$x = -y$

$x = 1, \quad y = -1$

$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is required eigen vector for $\lambda = 1$

(2) Find the eigen value & eigen vector of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

Sol The char. eqn of the given matrix is

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{bmatrix} = 0$$

It gives $\lambda = 1, -4, +7$

$AX = \lambda X$

Let $\lambda = 1$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, Then $AX = \lambda X$

or $(A - \lambda I)X = 0$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_2 + 3x_3 = 0$$
$$-5x_2 + 2x_3 = 0$$
$$6x_3 = 0$$

now solving we get $x_3 = 0$, $x_2 = 0$
x_1 is arbitrary (say k), hence the corresponding vector is $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

II When $\lambda = -4$
we get $-4X =$
$$AX = -4X$$
$$(A + 4I)X = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 = 0$$
$$11x_3 = 0$$

$\therefore$ $x_2 = x_3 = 0$ and taking $x_2 = k$ (arbit.)

$$5x_1 + 2k = 0$$
$$x_1 = -2k/5$$

hence the corresponding eigen vector

$$\begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix}$$

III when $\lambda = 7$, $AX = 7X$

$$(A - 7I)X = 0$$

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

taking x_3 as arbitrary

$$6x_1 + 2x_2 + 3x_3 = 0$$
$$-11x_2 + 2x_3 = 0$$

$$\therefore x_2 = 2k/11$$
$$x_1 = 37k/66$$

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Hence the corresponding eigen vector is

$$\begin{bmatrix} 37k/66 \\ 2k/11 \\ k \end{bmatrix}$$

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$x_1^2 + x_2^2 + x_3^2 = 1$

is termed as normalized or orthonormal vector. If $\lambda = 1$ then the corresponding eigen vector is $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ For this vector to be orthonormal, we must have

$k^2 + 0^2 + 0^2 = 1$ or $k = 1$ hence the corresponding orthonormal vector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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1. $|A-\lambda I|=0$ is called characteristics eqn of the matrix A

2. The roots of char. eqn $|A-\lambda I|=0$ are called char. roots of matrix A

$\lambda = 1, 1, 5$

e.g. $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$A-\lambda I = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \Rightarrow \text{char. matrix}$$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$ char. eqn

$\lambda = 1, 1, 5$, char. roots or eigen value

Properties of eigen ~~value~~ vector

① The eigen vector X of a matrix is not unique

② If $\lambda_1 \lambda_2 \dots \lambda_n$ ~~be~~ eigen values of $n \times n$ matrix then corresponding eigen vector $X_1 X_2 \dots X_n$ form a linearly independent set

③ $X_1' X_2 = 0$ orthogonal vectors

Non Symmetric matrices with non repeated eigen values

e.g. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Sol. $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = 1, 2, 3$$

To find eigen vectors for the corresponding eigen values we will consider the matrix equation

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -z=0 \\ x+y+z=0 \\ x+y+0=0 \end{array}$$

let $x=k$, $y=-k$

eigen vector $X_1 = \begin{bmatrix} k \\ -k \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

now putting $\lambda = 2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x-z=0 \\ x+z=0 \\ 2x+2y+z=0 \end{array}$$

eigen vector $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

now $\lambda = 3$,

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -2x-z=0 \\ x-y+z=0 \end{array}$$

eigen vector $X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Non Symm Matrix with Repeated eigen...
Find all eigen values and eigen ve...
of the matrix

$$A = \begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol. char. eqn of A is

$$\begin{bmatrix} -2-\lambda & 2 & 3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \quad \text{—} (1)$$

If $\lambda = -3$, $-27 + 9 + 63 - 45 = 0$

so $\lambda + 3$ is one factor of eqn (1)

the remaining factors ~~of~~ are obtained by dividing (1) from $\lambda + 3$

$$(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$$

$$\lambda = -3, -3, 5$$

To find the eigen vectors for correspond... eigen values

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -2-\lambda & 2 & 3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix}$$

on putting 5 in above eq^n

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x + 2y - 3z = 0$$
$$2x - 4y - 6z = 0$$
$$-x - 2y - 5z = 0$$

$$\Rightarrow x = 1, \quad y = 2, \quad z = -1$$

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - 3z = 0$$
$$2x + 4y - 6z = 0$$
$$-x - 2y + 3z = 0$$

first, second and third eq^n are same

$x = k_1, \quad y = k_2$

$$\therefore z = \frac{1}{3}(k_1 + 2k_2)$$

Let $k_1 = 0, \quad k_2 = 3$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Since the matrix is non symmetric the corresponding eigen vector X_2 & X_3 must be linearly independent. This can be done by choosing

$k_2 = 0$ & $k_1 = 1$

$$\therefore X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Symmetric Matrices with non repeated eigen values

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Sol. $[A - \lambda I] = 0$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

taking $\lambda = -3, -6, 12$

Matrix eq for eigenvectors $[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -3$

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+5y+4z=0 \\ 5x+10y+5z=0 \end{matrix}$$

$$\frac{x}{25-40} = \frac{y}{20-5} = \frac{z}{10-25}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

eigen vector corresponding to eigenvalue $\lambda = -6$

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$4x+5y+4z=0$$
$$5x+13y+5z=0$$

$$\frac{x}{25-52} = \frac{y}{20-20} = \frac{z}{52-25}$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

eigen vector corresponding to eigen value ... [illegible]

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $-14x + 5y + 4z = 0$

$5x - 5y + 5z = 0$

$$\frac{x}{25+20} = \frac{y}{20+70} = \frac{z}{70-25}$$

or $$\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Symmetric matrices with repeated eigenvalues

e.g. $$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Sol The chara. eqn is

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \text{—(1)}$$

on putting $\lambda = 1$ the eqn (1) is satisfi

so $\lambda - 1$ one factor of the eqn

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the other factor $(\lambda^2-5\lambda+4)$ is got on dividing (1) by $\lambda-1$

$(\lambda-1)(\lambda^2-5\lambda+4)=0$

$(\lambda-1)(\lambda-1)(\lambda-4)=0$

$\lambda=1,1,4 \Rightarrow$ eigen values

when $\lambda=4$

$$\begin{bmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

on interchanging first & third rows

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & -2 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

R_2+R_1, R_3+2R_1

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

R_3+R_2

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

$x_1-x_2-2x_3=0$

$-3x_2-3x_3=0$

Let $x_3=K$ $\quad -3x_2-3k=0 \quad \therefore x_2=-k$

Let $k=1$ $\quad x_1+k-2k=0 \quad \therefore x_1=K$

Scanned with CamScan

Let $k = 1$

The eigen vector $X_1' = (1, -1, 1)$

When $\lambda = 1$

$$x_1 + x_2 + x_3 = 0$$

Let $x_1 = k_1$ and $x_2 = k_2$

$k_1 - k_2 + x_3 = 0$ $\quad x_3 = k_2 - k_1$

eigen vector is $(k_1, k_2, k_2 - k_1)$

Choosing $k_1 = 1$, $k_2 = 1$

eigen vector $(1, 1, 0)$

Choosing $k_1 = -1$, $k_2 = 2$

eigen vector $X_2' = [1, 2, -1]'$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 & X_2

since the given matrix is sym

$[1, -1, 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ $\quad l - m + n = 0$

$[1, 1, 0] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ $\quad l + m + 0n = 0$

$$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{2}$$

$$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$