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Curve Fitting. Cubic Spline

If X & Y are two characteristics of the eqn, this functional relationship gives some observations to be drawn or to be predicted some value of Y for given X.

By plotting the given observations we can get some idea about the nature of the relationship between the two variables. The approximation can be in terms of polynomial eqn exponential, logarithmic, function, these functions contain certain unknown constants, which are to be determined with the help of given data or set of observation

Data Fitting with cubic splines

This is a subject of recent origin with important applications in numerical methods.

In the interpolation we have seen that a single polynomial has been fitted to given data. If the given set of points belong to the polynomial then the interpolation method give best approximation to the polynomial.

If we draw lines through every two neighbouring points, then the graph thus obtained will not smooth. on the other hand, if we draw a quardatic curve through two neighbouring points A_j, and A_{j+1} and another quadratic curve through A_{j+1}, & A_{j+2} such that two quadratic match at A_{j+1} then the resulting curves looks better but still it is not quite smooth.

Similarly two cubics are through

the slopes and curvatures of the two curves match q_{i+1}. Such curves are called cubic spline and plotting such curves are called spline fitting.

Cubic Spline Interpolation.

We consider now the problem of interpolating between given data points

(x_i, y_i) $i = 0, 1, 2 \ldots n$

where $a = x_0 < x_1 < x_2 \ldots < x_n = b$ by means of a smooth polynomial curve.

Let the smooth curve be a cubic spline $S(x)$, which satisfies the following properties

1. $S(x_i) = y_i$, for $i = 0, 1, 2 \ldots n$
2. $S(x)$, $S'(x)$, $S''(x)$ are continuous on $[a, b]$
3. $S(x)$ is a cubic polynomial in each sub interval $[x_i, x_{i+1}]$ for $i = 0, 1, 2, \ldots n-1$

In this spline theory the x_i are known as knots, and assuming these knots are arbitrary

$S(x)$ is a cubic spline in (x_i, x_{i+1}) then $S''(x)$ must be linear and hence it can be written as

$$S''(x) = \frac{1}{h_i}\left[(x_{i+1} - x)\, S''(x_i) + (x - x_i)\, S''(x_{i+1})\right]$$

where $h_i = x_{i+1} - x_i$

integrating above eqn two time

$$S(x) = \frac{1}{h_i}\; \frac{(x_{i+1} - x)^3}{3!} \cdot S''(x_i) + \frac{(x - x_i)^3}{3!}$$

$$S''(x_{i+1}) + C_1(x_{i+1} - x) + C_2(x - x_i) \quad \text{--- (2)}$$

C_1 & C_2 → Constants, are to be determined, using the condition 1

$$S(x_i) = y_i$$

$$y_i = \frac{1}{h_i}\left[\frac{h_i^3}{6} S''(x_i)\right] + C_1 h_i$$

$$\therefore C_1 = \frac{y_i}{h_i} - \frac{h_i}{6} S''(x_i)$$

Also $S(x_{i+1}) = y_{i+1}$

$$y_{i+1} = \frac{1}{h_i}\left[\frac{h_i^3}{6} S''(x_{i+1}) + C_2 h_i\right]$$

$$\therefore C_2 = \frac{y_{i+1}}{h_i} - \frac{h_i}{6} S''(x_{i+1})$$

substituting value of C_1 & C_2 in (2)

$$S(x) = \frac{1}{h_i}\left[\frac{(x_{i+1} - x)^3}{6} S''(x_i) + \frac{(x - x_i)^3}{6} S''(x_{i+1})\right]$$

Page No.:
Date:

$$+\frac{1}{h_i}(x_{i+1}-x)\left[y_i - \frac{h_i^2}{6} S''(x_{i+1})\right] + \frac{1}{h_i}(x-x_i)$$

$$\left[y_{i+1} - \frac{h_i^2}{6} S''(x_{i+1})\right] \quad \text{-(1)}$$

Now substituting $S''(x_i) = M_i$

$S''(x_{i+1}) = M_{i+1}$

$$S(x) = \frac{1}{6h_i}\left[(x_{i+1}-x)^3 M_i + (x-x_i)^2 M_{i+1}\right] + \frac{1}{h_i}$$

$$(x_{i+1}-x)\left[y_i - \frac{h_i^2}{6}M_i\right] + \frac{1}{h_i}(x-x_i)\left[\right.$$

$$\left.y_{i+1} - \frac{h_i^2}{6} M_{i+1}\right] \quad \text{- - - (3)}$$

$$\therefore S'(x) = \frac{1}{6h_i}\left[-3(x_{i+1}-x)^2 M_i + 3(x-x_i)^2 M_{i+1}\right]$$

$$-\frac{1}{h_i}\left[y_i - \frac{h_i^2}{6}M_i\right] + \frac{1}{h_i}\left[y_{i+1} - \frac{h_i^2}{6}M_{i+1}\right]$$

$S'(x)$ is continuous at $x = x_i$, then

$$S'(x_i+0) = S'(x_i-0) \quad \text{- - (4)}$$

so from eq^n (3) we obtain

$$S'(x_i+0) = -\cancel{\frac{h^2}{h}} - \frac{h_i}{6}(2M_i + M_{i+1}) +$$

$$\frac{2}{h_i}(y_{i+1} - y_i) \quad \text{- - (5)}$$

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Page No.:
Date:

Again in interval $[x_{i-1}, x_i]$ the spline after differentiating

$$S'(x_i - 0) = \frac{h_{i-1}}{6}\left[2M_i + M_{i-1}\right] + \frac{1}{h_{i-1}}(y_i - y_{i-1})$$

Now from eqn 4, 5, 6 we get

$$h_{i-1} M_{i-1} + 2(h_{i-1} + h_i) M_i + h_i M_{i+1} =$$

$$6\left[\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_i}\right] \ldots$$

For equal interval we have
$h_{i-1} = h_i = h$ then eqn (7) becomes

$$\boxed{h^2\left[M_{i-1} + 4M_i + M_{i+1}\right] = 6\left[y_{i-1} - 2y_i + y_{i+1}\right]}$$

where $i = 1, 2 \ldots n-1$ Formula

Eqn (8) gives a system of $(n-1)$ linear equations in $(n+1)$ unknowns $M_0, M_1 \ldots M_n$ substituting the value of M_i in (3) we get required cubic spline. When M_i are known the eqn (5) + (6) can be used to obtain approximation for y_i'

End conditions – There are ~~no~~ several ways in which the end conditions may be imposed so that different types of cubic splines are obtained

1. If $M_0 = M_n = 0$ then the spline is called natural cubic spline

2. If $M_0 = M_n$, $M_1 = M_{n+1}$

$y_0 = y_n$

$y_1 = y_{n+1}$

$h_1 = h_{n+1}$ then the spline is known as periodic spline.

(3) If $S'(a) = y'(a) = y'(0)$

and $S'(b) = y'(b)$ $y'(n)$ then the spline is non periodic

4. If $M_0 = M_1$ & $M_{n-1} - M_n$ then the spline in the end intervals are parabolas.

Ex 1 The following values of x & y are given

x:	1	2	3	4
y:	1	2	5	11

Find the cubic spline and evaluat $y(1.5)$ and $y'(3)$

Soln Here the points x_0, x_1, x_2, x_3 are equispaced with $h=1$ and $n=3$ then we have

$$h^2(M_{i-1} + 4M_i + M_{i+1}) = 6(y_{i-1} - 2y_i + y_{i+1})$$

for $i = 1, 2$

$x_0 = 1,$

Page No.:
Date:

$i = 1$

$$(M_0 + 4M_1 + M_2) = 6(y_0 - 2y_1 + y_2)$$
$$= 6(1 - 2(2) - 5) = 12$$

Where from table $y_0 = 1,\ y_1 = 2,\ y_2 = 5$

$i = 2$

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$$
$$= 6(2 - 2(5) + 11) = 18$$

where $y_1 = 2,\ y = 5,\ y_3 = 11$

Since the begining graph is linear

$M_0 + 4M_1 + M_2 = 12$

$M_1 + 4M_2 + M_3 = 18$

we only taken $i = 1, 2$ so

$M_0 = 0,\quad M_3 = 0$

$\therefore\ 4M_1 + M_2 = 12$

$M_1 + 4M_2 = 18$

$\Rightarrow\ M_1 = 2,\quad M_2 = 4.$

Hence the cubic splines are

$S(x) =\quad x_0 = 1,\ x_1 = 2\quad x_2 = 3,\ x_3 = 4$

$\quad y_0 = 1\quad y_1 = 2\quad y_2 = 5\quad y_3 = 11$

$$S(x) = \frac{1}{6h}\left[(x_{i+1} - x)^3 M_i (x - x_i)^3 M_{i+1}\right] + \frac{1}{h}(x_{i+1} - x)\left[y_i - \frac{h^2}{6}M_i\right] + \frac{1}{h}(x - x_i)\left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right)$$

cubic spline in the interval $1 \le x \le 2$

(x_1, x_2)

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Page No.:
Date:

$S(x) = \frac{1}{6}\left[(x_2-x)^3 M_1 + (x-x_1)^3 M_2\right] + \frac{1}{1}\left[x_2-x\right]\left[y_1 - \frac{1}{6}M_1\right]$
$+ \frac{1}{1}\left[x-x_1\right]\left(y_2 - \frac{1}{6}M_2\right)$

$= \frac{1}{6}\left[(3-x)^3 2 (x-2)^3 4\right] + \left[3-x\right]\left[2-\frac{1}{6}\cdot 2\right] +$
$\left[x-2\right]\left[5-\frac{4}{6}\right]$

$= \frac{1}{6}\left[(3-x)^3 2 (x-2)^3 4 + (3-x)\left(\frac{5}{6}\right) + (x-2)\left[\frac{26}{6}\right]\right]$

$= \frac{1}{3}\left(x^3 - 3x^2 + 5x\right)$

$= \frac{1}{3}\left(x^3 - 3x^2 + 5x\right)$ for $2 \le x \le 3$

$= \frac{1}{3}\left(-2x^3 + 24x^2 - 76x + 81\right)$ $3 \le x \le 4$

$\therefore y(1.5) = f(1.5)$
$= \frac{1}{3}(1.5)^3 - 3(1.5)^2 + 5(1.5) = \frac{11}{8}$

$y'(x) = \frac{1}{3}\left(-6x^2 + 48x - 76\right)$ $(3,4)$

$y'(3) = \frac{1}{3}\left(-6(3)^2 + 48(3) - 76\right) = 14/3$

Ans $y(1.5) = 11/8$
$y'(3) = 14/3$